

3.6.1 Constructing continuous functions

Theorem 3.64. 1. (Constant function) If $f : X \rightarrow Y$ maps all of X to a single point $y_0 \in Y$, then f is continuous.

2. (Inclusion) If A is a subspace of X , the inclusion map $i : A \rightarrow X$ defined by $i(a) = a$ is continuous.

3. (Composition) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the composition $g \circ f : X \rightarrow Z$ is continuous.

4. (Restricting the domain) If $f : X \rightarrow Y$ is continuous and $A \subseteq X$, then the restriction $f|_A : A \rightarrow Y$ defined by $f|_A(a) = f(a)$ is continuous.

5. (Restricting or expanding the range) Let $f : X \rightarrow Y$ be continuous. If $f(X) \subset Z \subset Y$, then the function $g : X \rightarrow Z$ obtained by restricting the range of f is continuous. If $Y \subset W$, then the function $h : i_{Y \rightarrow W} \circ f : X \rightarrow W$ is continuous.

6. (Local formulation) $f : X \rightarrow Y$ is continuous if $X = \cup U_\alpha$ for a family of open sets U_α and each restriction $f|_{U_\alpha}$ is continuous.

7. (Pasting lemma) If $X = A \cup B$ with A, B closed, and $f : A \rightarrow Y$, $g : B \rightarrow Y$ are continuous and agree on $A \cap B$ (i.e., $f(x) = g(x)$ for all $x \in A \cap B$), then the function $h : X \rightarrow Y$ defined by

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is continuous.

Proof. 1. For any open set U in Y , if $y_0 \in U$, then $f^{-1}(U) = X$; otherwise, $f^{-1}(U) = \emptyset$. In both cases, $f^{-1}(U)$ is open in X . Hence, f is continuous.

2. For any open set U in X , $i^{-1}(U) = U \cap A$ is open in A . Hence, i is continuous.

3. For any open set W in Z , $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since g is continuous, $g^{-1}(W)$ is open in Y . Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X . Hence, $g \circ f$ is continuous.

4. For any open set U in Y , $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$ is open in A . Hence, $f|_A$ is continuous.

5. For any open set V in Z , then there exists U open in Y such that $V = U \cap Z$. Hence, $g^{-1}(V) = f^{-1}(V) = f^{-1}(U \cap Z)$. Since $f(X) \subset Z$, we have that $f^{-1}(U \cap Z) = f^{-1}(U)$ is open in X . Hence, g is continuous. The continuity of h follows from 2 and 3.

6. For any open set U in Y , we first note that

$$f^{-1}(U) \cap U_\alpha = (f|_{U_\alpha})^{-1}(U) = \{x \in U_\alpha : f(x) \in U\}.$$

Since $f|_{U_\alpha}$ is continuous, $(f|_{U_\alpha})^{-1}(U)$ is open in U_α . Hence,

$$f^{-1}(U) = \bigcup (f^{-1}(U) \cap U_\alpha) = \bigcup (f|_{U_\alpha})^{-1}(U)$$

is open in X . This shows that f is continuous.

7. For any closed set C in Y , $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since A and B are closed, $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X (see the later lemma for this). Hence, $h^{-1}(C)$ is closed in X . This shows that h is continuous.

Lemma 3.65. *Let $A \subset X$ and let $E \subset A$. Then, E is closed in A if and only if there exists a closed set F in X such that $E = F \cap A$.*

Proof. Suppose E is closed in A . Then $A \setminus E$ is open in A , so there exists an open set U in X such that $A \setminus E = U \cap A$. Let $F = X \setminus U$, which is closed in X . Then

$$F \cap A = (X \setminus U) \cap A = A \setminus (U \cap A) = A \setminus (A \setminus E) = E.$$

Conversely, if $E = F \cap A$ for some closed set F in X , then

$$A \setminus E = A \setminus (F \cap A) = A \cap (X \setminus F),$$

where $X \setminus F$ is open in X . Hence, $A \setminus E$ is open in A , so E is closed in A . $\square$

Corollary 3.66. *Let $A \subset X$ and let $E \subset A$. If E is closed in A then E is closed in X .*

Proof. Since E is closed in A , there exists a closed set F in X such that $E = F \cap A$. Since $A \subset X$ is closed, we have that E is closed in X $\square$

$\square$

Remark 3.67. 7 is the closed set and finite version of 6. See your homework for the infinite version of 7.

Theorem 3.68. *Let $f : A \rightarrow X \times Y$ be given by $f(a) = (f_X(a), f_Y(a))$. Then f is continuous if and only if f_X and f_Y are continuous.*

Proof. Recall the projections $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$. Note that $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$.

$\Rightarrow$: If f is continuous, then f_X and f_Y are continuous by the composition of continuous functions.

$\Leftarrow$: For $U \times V$ a basis element in $X \times Y$, we have that

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)) \\ &= f^{-1}(\pi_X^{-1}(U)) \cap f^{-1}(\pi_Y^{-1}(V)) \\ &= f_X^{-1}(U) \cap f_Y^{-1}(V). \end{aligned}$$

Since f_X and f_Y are continuous, $f^{-1}(U \times V)$ is open in A . This shows that f is continuous. $\square$

3.6.2 Homeomorphism

Definition 3.69. A function $f : X \rightarrow Y$ between topological spaces is a *homeomorphism* if it is a bijection and both f and f^{-1} are continuous.

If f is not surjective, we let $Z = f(X)$. If the restriction $f' : X \rightarrow Z$ is a homeomorphism, then we say that f is a topological embedding of X into Y .

Homeomorphisms preserve topological properties, meaning that if X and Y are homeomorphic, they share all topological characteristics. More precisely, we have the following theorem.

Theorem 3.70. *Let $f : X \rightarrow Y$ be a homeomorphism. Then, f induces a map $f_* : \tau_X \rightarrow \tau_Y$ defined by $f_*(U) = f(U)$ for all $U \in \tau_X$. Furthermore, f_* is a bijection.*

Proof. For any $U \in \tau_X$, $f(U) = (f^{-1})^{-1}(U)$. Since f^{-1} is continuous, $f(U)$ is open in Y . This shows that f_* is well-defined.

Next, we show that f_* is injective. If $f_*(U_1) = f_*(U_2)$ for some $U_1, U_2 \in \tau_X$, then $f(U_1) = f(U_2)$. Since f is bijective, we have that $U_1 = U_2$. This shows that f_* is injective.

Finally, we show that f_* is surjective. For any $V \in \tau_Y$, since f^{-1} is continuous, we have that $f^{-1}(V)$ is open in X . Then,

$$f_*(f^{-1}(V)) = f(f^{-1}(V)) = V.$$

This shows that f_* is surjective.

Therefore, we conclude that $f_* : \tau_X \rightarrow \tau_Y$ is a bijection. $\square$

Example 3.71. Suppose you have a subspace in some ambient Euclidean space. You can stretch and bend it without tearing or gluing. This is a homeomorphism. For example, a coffee cup is homeomorphic to a donut (solid torus).

Example 3.72. The circle and the trefoil knot are homeomorphic. But we can't deform one to the other in 3D space without cutting.

Example 3.73. The map $f : (-1, 1) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x}{1-x^2}$ is a homeomorphism with inverse $f^{-1} : \mathbb{R} \rightarrow (-1, 1)$ given by $f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}$. So homeomorphisms need not preserve boundedness.

Example 3.74 (Continuous bijection not a homeomorphism). Let

$$X = [0, 1)$$

with the usual topology from $\mathbb{R}$, and

$$Y = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

with the subspace topology from $\mathbb{R}^2$.

Define

$$f : X \rightarrow Y, \quad f(t) = (\cos 2\pi t, \sin 2\pi t).$$

Then f is a continuous bijection. However, f^{-1} is not continuous: around $(1, 0) \in S^1$, neighborhoods correspond to t near 0 and near 1, which are far apart in $[0, 1)$, so f^{-1} fails continuity there.

This shows that a continuous bijection need not be a homeomorphism when the domain is not compact.

Example 3.75. Is $(0, 1)$ homeomorphic to $[0, 1)$?

Lemma 3.76. Let $f : X \rightarrow Y$ be a homeomorphism. Let $x_0 \in X$. Then, the map $g : X \setminus \{x_0\} \rightarrow Y \setminus \{f(x_0)\}$ defined by $g(x) = f(x)$ is also a homeomorphism.

Definition 3.77. Let X be a topological space. We say X is connected if it cannot be represented as the union of two disjoint non-empty open sets.

Remark 3.78. If X is connected and X is homeomorphic to Y , then Y is also connected. This holds obviously as the homeomorphism gives rise to a bijection between the topologies.

Now we are ready to answer the previous question.

Theorem 3.79. $(0, 1)$ is not homeomorphic to $[0, 1)$.

Proof. Suppose there exists a homeomorphism $f : [0, 1) \rightarrow (0, 1)$. Pick any $x_0 = 0$ and consider the map $g : (0, 1) \rightarrow (0, 1) \setminus \{f(x_0)\}$ defined by $g(x) = f(x)$. By the previous lemma, g is a homeomorphism. Note that $(0, 1) \setminus \{f(x_0)\}$ is not connected regardless the value of $f(x_0)$. However, $(0, 1) = [0, 1) \setminus \{x_0\}$ is connected. This is a contradiction. $\square$

Example 3.80. Similarly, we can prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

How about $\mathbb{R}^2$ vs $\mathbb{R}^3$? Need algebraic topology. We will have a taste of it later in the course.

3.7 Product Topology

Recall the Cartesian product of a family of sets $\{X_i\}_{i \in I}$ is the set of functions:

$$\prod_{i \in I} X_i := \{x : I \rightarrow \cup_{i \in I} X_i \mid \text{for all } i \in I, x(i) \in X_i\}.$$

Let's mimic the case when I is a two-point set to identify the following definition of topology.

Definition 3.81 (Box topology (less important)). For spaces (X_i, τ_i) , $i \in I$, the *box topology* on $\prod_{i \in I} X_i$ is generated by the basis

$$\mathcal{B}_{\text{box}} = \left\{ \prod_{i \in I} U_i : U_i \in \tau_i \text{ for every } i \right\}$$

Let's first verify that $\mathcal{B}_{\text{box}}$ is indeed a basis. For any $x \in \prod_{i \in I} X_i$, we have that $x \in \prod_{i \in I} X_i = \prod_{i \in I} X_i$. Furthermore, for any two basis elements $\prod_{i \in I} U_i$ and $\prod_{i \in I} V_i$, we have that

$$x \in \prod_{i \in I} (U_i \cap V_i) = \prod_{i \in I} U_i \cap \prod_{i \in I} V_i.$$

Since U_i and V_i are open in X_i , $U_i \cap V_i$ is open in X_i . This shows that $\mathcal{B}_{\text{box}}$ is indeed a basis.

This basis seems reasonable right? But this topology is too fine and makes many things not work.

Example 3.82. Let $X = \mathbb{R}^{\mathbb{N}} = \prod_{n \geq 1} \mathbb{R}$. Then, the “diagonal” map $f : \mathbb{R} \rightarrow X$ defined by $f(t) = (t, t, t, \dots)$ is not continuous when X is endowed with the box topology. Consider the open set $(-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots$. Then,

$$\begin{aligned} & f^{-1}((-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots) \\ &= f^{-1}((-1, 1) \times \mathbb{R} \times \dots \cap \mathbb{R} \times (-1/2, 1/2) \times \mathbb{R} \times \dots \cap \dots) \\ &= \bigcap_{n \geq 1} f^{-1}(\mathbb{R} \times \dots \times (-1/n, 1/n) \times \dots) \\ &= \bigcap_{n \geq 1} (-1/n, 1/n) = \{0\}. \end{aligned}$$

Then $f^{-1}((-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \dots)$ is not open in $\mathbb{R}$. Hence, f is not continuous.

This example actually gives us a hint how we can make things right. The problem above is that when considering the arbitrary intersection of open sets, we don't get an open set. SO, we want things to be finite in some sense. This leads to the following definition.

Definition 3.83 (Product topology). For spaces (X_i, τ_i) , $i \in I$, the *product* $X = \prod_{i \in I} X_i$ carries the product topology generated by the basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_{i \in I} U_i : U_i \in \tau_i \text{ and } U_i = X_i \text{ for all but finitely many } i \right\}.$$

Now we verify that $\mathcal{B}_{\text{prod}}$ is indeed a basis. For any $x \in \prod_{i \in I} X_i$, we have that $x \in \prod_{i \in I} X_i = \prod_{i \in I} X_i$. Furthermore, for any two basis elements $\prod_{i \in I} U_i$ and $\prod_{i \in I} V_i$, we have that

$$\prod_{i \in I} (U_i \cap V_i) = \prod_{i \in I} U_i \cap \prod_{i \in I} V_i.$$

Since U_i and V_i are open in X_i , $U_i \cap V_i$ is open in X_i . Furthermore, since $U_i = X_i$ for all but finitely many i and $V_i = X_i$ for all but finitely many i , we have that $U_i \cap V_i = X_i$ for all but finitely many i . This shows that $\mathcal{B}_{\text{prod}}$ is indeed a basis.

Remark 3.84. Since $\mathcal{B}_{\text{prod}} \subseteq \mathcal{B}_{\text{box}}$, the product topology is coarser than the box topology. If the index set I is finite, then the box topology and the product topology coincide.

Remark 3.85. The identity map $(\prod_{i \in I} X_i, \tau_{\text{box}}) \rightarrow (\prod_{i \in I} X_i, \tau_{\text{prod}})$ is continuous; the inverse need not be when I is infinite.

Example 3.86. Now we revisit this example where $X = \mathbb{R}^{\mathbb{N}} = \prod_{n \geq 1} \mathbb{R}$. Then, the “diagonal” map $f : \mathbb{R} \rightarrow X$ defined by $f(t) = (t, t, t, \dots)$ is continuous when X is endowed with the product topology. Consider any basis element $U = \prod_{n \geq 1} U_n$ in X . Then, there exists $N > 0$ such that for all $n > N$, $U_n = \mathbb{R}$. Then,

$$\begin{aligned} f^{-1}\left(\prod_{n \geq 1} U_n\right) &= f^{-1}\left(\prod_{n=1}^N U_n \times \prod_{n > N} \mathbb{R}\right) \\ &= f^{-1}\left(\bigcap_{n=1}^N \mathbb{R} \times \cdots \times U_n \times \mathbb{R} \times \cdots\right) \\ &= \bigcap_{n=1}^N f^{-1}(\mathbb{R} \times \cdots \times U_n \times \mathbb{R} \times \cdots) \\ &= \bigcap_{n=1}^N U_n. \end{aligned}$$

Since (U_n) is open in $\mathbb{R}$ for all n , $f^{-1}(\prod_{n \geq 1} U_n)$ is open in $\mathbb{R}$. Hence, f is continuous.

Example 3.87. Let $X = \mathbb{R}^{\mathbb{N}} = \prod_{n \geq 1} \mathbb{R}$. Then $(0, 1)^{\mathbb{N}}$ is box open but not product open (it contains no basis element).

The product topology has a simple subbasis.

Theorem 3.88. *The product topology on $X = \prod_{i \in I} X_i$ is generated by the subbasis*

$$\mathcal{S} = \{\pi_i^{-1}(U_i) : i \in I, U_i \text{ open in } X_i\}$$

where $\pi_i : X \rightarrow X_i$ is the projection onto the i th factor.

Proof. Note that for any $i \in I$ and any open set U_i in X_i , $\pi_i^{-1}(U_i)$ is of the form

$$\pi_i^{-1}(U_i) = \prod_{j \in I} V_j$$

where $V_j = U_i$ if $j = i$ and $V_j = X_j$ otherwise.

So, for any finite family of such sets $\pi_{i_1}^{-1}(U_{i_1}), \dots, \pi_{i_n}^{-1}(U_{i_n})$, we have that

$$\bigcap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k}) = \prod_{j \in I} W_j$$

where $W_j = U_{i_k}$ if $j = i_k$ for some $k \in \{1, \dots, n\}$ and $W_j = X_j$ otherwise. This is a basis element in the product topology.

Conversely, it is easy to see that any basis element $\prod_{i \in I} U_i$ in the product topology can be written as a finite intersection of sets in $\mathcal{S}$ as above. $\square$

Here is a one great example showing why people care about the product topology.

Definition 3.89. Let $\pi_i : \prod_{j \in I} X_j \rightarrow X_i$ be the projection onto the i th factor, i.e., $\pi_i(x) = x(i)$.