

Theorem 3.90 (Minimality). *Let τ_Π be the product topology on $X = \prod_{i \in I} X_i$. It is the smallest topology on X making every projection $\pi_i : X \rightarrow X_i$ continuous: if τ is any topology on X with all π_i continuous, then $\tau_\Pi \subseteq \tau$.*

Proof. For any U_i open in X_i , we have that $\pi_i^{-1}(U_i)$ must be contained in τ . So $\tau \supset \mathcal{S}$ and hence $\tau \supset \tau_\Pi$. $\square$

Theorem 3.91. *Consider any function $f : Y \rightarrow \prod_{j \in I} X_j$. Then f is continuous iff each coordinate projection $f_i := \pi_i \circ f : Y \rightarrow X_i$ is continuous.*

Proof. Note each projection π_i is continuous by the theorem above.

$\Rightarrow$: If f is continuous, then $f_i = \pi_i \circ f$ is continuous by the composition of continuous functions.

$\Leftarrow$: For any subbasis element $\pi_i^{-1}(U_i)$ where U_i is open in X_i , we have that $f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i) = f_i^{-1}(U_i)$ is open in Y since f_i is continuous. This proves that f is continuous. $\square$

Remark 3.92. The $\Leftarrow$ direction needs not be true if we consider the box topology instead of the product topology. The same diagonal map $f : \mathbb{R} \rightarrow \mathbb{R}^\mathbb{N}$ defined by $f(t) = (t, t, t, \dots)$ is a counterexample: every $f_i(t) = t$ is continuous but f is not continuous when $\mathbb{R}^\mathbb{N}$ is endowed with the box topology.

Now let's talk about the product of metric spaces. We have one example in homework 3 but we will give a more general construction here.

Lemma 3.93. *Let (X, d) be a metric space. Define $d' : X \times X \rightarrow \mathbb{R}$ by*

$$d'(x, y) = \min\{d(x, y), 1\}.$$

Then, d' is a metric on X and the topology induced by d' is the same as the topology induced by d .

Proof. It is easy to check that d' is a metric.

Since $d'(x, y) \leq d(x, y)$, then $\tau_{d'} \subset \tau_d$.

Now, we show that $\tau_d \subset \tau_{d'}$. For any $x \in X$ and any $r > 0$, if $r \leq 1$, then $B_d(x, r) = B_{d'}(x, r)$; if $r > 1$, then $B_d(x, r) \supset B_{d'}(x, 1)$. This shows that $\tau_d \subset \tau_{d'}$. $\square$

Theorem 3.94. *Let $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ be a family of metric spaces. Let $X = \prod_{i \in \mathbb{N}} X_i$. For any $i \in \mathbb{N}$, we define $d'_i : X \times X \rightarrow \mathbb{R}$ by*

$$d'_i(x, y) = \min\{d_i(x(i), y(i)), 1\}.$$

Then, the function $d : X \times X \rightarrow \mathbb{R}$ defined by

$$d(x, y) = \sum_{i \in \mathbb{N}} \frac{d'_i(x, y)}{2^i}$$

is a metric on X and the topology induced by d is the product topology.

The proof is similar to Problem 2 in Homework 4 and is omitted here.

Example 3.95. $\mathbb{R}^{\mathbb{N}}$ with the box topology is not metrizable. Let

$$A = \{(x_1, \dots) | x_i > 0\}$$

and let $\mathbf{0} = (0, 0, \dots)$. Then, $\mathbf{0} \in \overline{A}$: for any basis element

$$U = (a_1, b_1) \times (a_2, b_2) \times \dots$$

containing $\mathbf{0}$, we have that $a_i < 0 < b_i$ for all i . So pick any point $(x_1, x_2, \dots)$ with $x_i \in (0, b_i)$ for all i , then $(x_1, x_2, \dots) \in A \cap U$. So, $\overline{A} \cap U \neq \emptyset$.

Assume that $\mathbb{R}^{\mathbb{N}}$ with the box topology is metrizable with a compatible metric d . Let $\mathbf{a}_n = (x_{1n}, x_{2n}, \dots)$ be a sequence in A converging to $\mathbf{0}$. Then, construct $B = \{(-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots\}$. Then, $\mathbf{0} \in B$ but $\mathbf{a}_n \notin B$ for all $n \in \mathbb{N}$. So there exists a ball $B_r(\mathbf{0}) \subseteq B$ for some $r > 0$ containing no point in the sequence $\mathbf{a}_n$. This is a contradiction. So $\mathbb{R}^{\mathbb{N}}$ with the box topology is not metrizable.

3.8 Quotient spaces

We have seen how to construct topologies on various sets. One way to create a new space from old ones is via gluing. More precisely, let X be a topological space and let X^* be a partition of X (in other words, an equivalence relation $\sim$ on X so that $X^* = X/\sim$).

Example 3.96. One can obtain a torus and a sphere via gluing/equivalence relations.

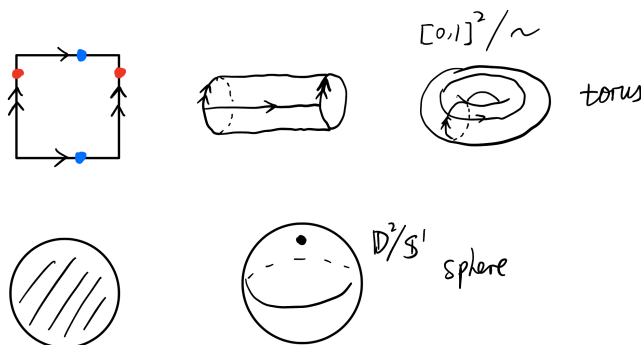


Figure 1: A torus and a sphere, illustrating the concept of quotient spaces.

How can we get a topology on X^* from the topology on X ? Recall that the product topology can be characterized via some continuous maps. Maybe we can do something similar here. A very natural map is the quotient map $p : X \rightarrow X^*$ sending each point to its equivalence class.

We want the finest topology on X^* that makes p continuous. Thus we declare a subset $U \subseteq X^*$ to be open whenever $p^{-1}(U)$ is open in X . This includes every subset that can be open without violating continuity of p .

Definition 3.97 (Quotient topology). Let $q : X \rightarrow Y$ be a surjective map. The *quotient topology* on Y is defined by

$$\tau_q := \{V \subseteq Y \mid q^{-1}(V) \text{ is open in } X\}$$

When endowed with the quotient topology, we call Y a *quotient space* of X and q a *quotient map*.

Check: this is indeed a topology. $q^{-1}(Y) = X$ and $q^{-1}(\emptyset) = \emptyset$. So $Y, \emptyset \in \tau_q$.

$$q^{-1}(\bigcup_{\alpha} V_{\alpha}) = \bigcup_{\alpha} q^{-1}(V_{\alpha}), \text{ so } \bigcup_{\alpha} V_{\alpha} \in \tau_q.$$

$$q^{-1}(\bigcap_{i=1}^n V_i) = \bigcap_{i=1}^n q^{-1}(V_i), \text{ so } \bigcap_{i=1}^n V_i \in \tau_q.$$

□

Remark 3.98. The quotient topology is the finest topology on Y making q continuous: if τ is any topology on Y with q continuous, then $\tau \subseteq \tau_q$.

Definition 3.99 (Quotient space). Let X be a topological space. Let $\sim$ be an equivalence relation on X and let $X^* := X / \sim$ be the set of equivalence classes. The *quotient space* of X by $\sim$ is the set X^* endowed with the quotient topology via the natural projection map $p : X \rightarrow X^*$ sending each point to its equivalence class.

Example 3.100 (A 3-point quotient of $\mathbb{R}$). Let $p : \mathbb{R} \rightarrow Y = \{-1, 0, 1\}$ be

$$p(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Endow Y with the quotient topology via p . A subset $V \subseteq Y$ is open iff $p^{-1}(V)$ is open in $\mathbb{R}$.

- $p^{-1}(\{-1\}) = (-\infty, 0)$ and $p^{-1}(\{1\}) = (0, \infty)$ are open $\Rightarrow \{-1\}, \{1\}$ are open.
- $p^{-1}(\{0\}) = \{0\}$ is not open $\Rightarrow \{0\}$ not open.
- $p^{-1}(\{-1, 1\}) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is open $\Rightarrow \{-1, 1\}$ open.
- $p^{-1}(\{-1, 0\}) = (-\infty, 0]$ and $p^{-1}(\{0, 1\}) = [0, \infty)$ are not open.

Thus

$$\tau_Y = \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}, \{-1, 0, 1\}\}.$$

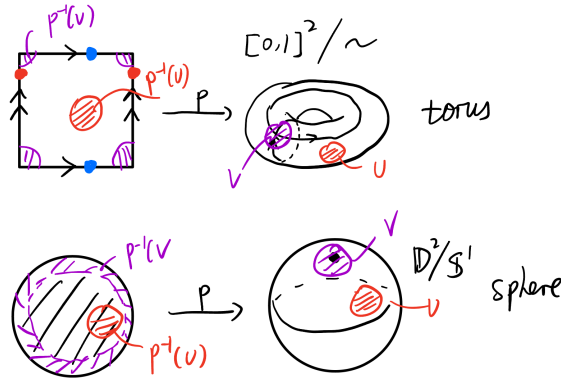


Figure 2: A torus and a sphere, illustrating the concept of quotient spaces.

Example 3.101 (Torus and Sphere). Consider the maps shown in the image (Figure 2). We then can endow the torus and the sphere (viewed as subsets of $\mathbb{R}^3$) with the quotient topologies via these maps. Of course, there is the question whether these quotient topologies coincide with the subspace topologies from $\mathbb{R}^3$. We will answer this question later.

Now let $q : X \rightarrow Y$ be surjective. What if we have another topology τ on Y ?

Definition 3.102. For two topological spaces X, Y and a surjective map $q : X \rightarrow Y$, we say q is a *quotient map* if

a subset $U \subseteq Y$ is open in Y iff $q^{-1}(U)$ is open in X .

In other words, q is a quotient map if the topology on Y coincides with the quotient topology via q .

Whenever we have a quotient map $q : X \rightarrow Y$, we can identify Y with the quotient space X^* of X by the equivalence relation induced by q . More precisely, we have the following proposition.

Proposition 3.103. Let $q : X \rightarrow Y$ be a quotient map. Define $\sim$ on X by $x \sim x'$ iff $q(x) = q(x')$ and let $X^* := X / \sim = \{q^{-1}(y) : y \in Y\}$ endowed with the quotient topology. Then, q induces a homeomorphism $\varphi : X^* \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X^* \\ & \searrow q & \downarrow \varphi \\ & & Y \end{array}$$

Proof. We first define $\varphi : X^* \rightarrow Y$ by $\varphi(q^{-1}(y)) = y$. This is well-defined since $q^{-1}(y) = q^{-1}(y')$ implies $y = y'$. It is easy to see that φ is bijective and the diagram commutes.

Next, we show that φ is continuous. For any open set $U \subseteq Y$, we have that

$$p^{-1}(\varphi^{-1}(U)) = q^{-1}(U)$$

is open since q is continuous. By the definition of the quotient topology, $\varphi^{-1}(U)$ is open in X^* and hence φ is continuous.

Finally, we show that φ^{-1} is continuous. For any open set $V \subseteq X^*$, we have that

$$q^{-1}(\varphi(V)) = p^{-1}(V)$$

is open since p is continuous. By the definition of the quotient map, $\varphi(V)$ is open in Y and hence φ^{-1} is continuous. $\square$

Here is some simple criterion for quotient maps. A map is said to be *open* if the image of every open set is open. Similarly, a map is said to be *closed* if the image of every closed set is closed.

Theorem 3.104. Let $q : X \rightarrow Y$ be a surjective continuous map. If q is open or closed, then q is a quotient map.

Proof. We only need to show the “only if” direction. Suppose q is open. Let $U \subseteq Y$ be such that $q^{-1}(U)$ (we used surjection here) is open in X . Then, $U = q(q^{-1}(U))$ is open in Y since q is open. The case when q is closed is similar. $\square$

Example 3.105 (A quotient map that is neither open nor closed). Let $A = [0, 1) \subset \mathbb{R}$. Identify all points of A with one another and leave every other point distinct: define $x \sim y$ if $x = y$ or both $x, y \in A$. Give $Y = \mathbb{R}/\sim$ the quotient topology and let $q : \mathbb{R} \rightarrow Y$ be the natural projection. Then q is a quotient map.

Write $*$ for the point corresponding to A . Since

$$q^{-1}(\{*\}) = [0, 1)$$

is neither open nor closed in $\mathbb{R}$, the singleton $\{*\}$ is neither open nor closed in Y . But $q((0, 1)) = \{*\}$ although $(0, 1)$ is open, and $q(\{0\}) = \{*\}$ although $\{0\}$ is closed. Thus q is neither an open map nor a closed map, so the converse of the preceding theorem fails.

Example 3.106. Consider the subspace

$$X = \{(x, y) \in \mathbb{R}^2 : xy = 1\} \cup \{(0, 0)\} \subset \mathbb{R}^2, \quad q : X \longrightarrow \mathbb{R}, \quad q(x, y) = x.$$

The map q is surjective: for $x \neq 0$ choose $y = 1/x$, and $q(0, 0) = 0$. It is continuous as the restriction of the projection $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$.

However q is not a quotient map. Indeed $\{(0, 0)\} = B_\varepsilon(0, 0) \cap X$ for small $\varepsilon > 0$, so $\{(0, 0)\}$ is open in X . Thus

$$q^{-1}(\{0\}) = \{(0, 0)\} \text{ is open in } X,$$

but $\{0\}$ is not open in $\mathbb{R}$. Thus q is continuous and surjective, but not a quotient map.

Example 3.107. Let $X = [0, 2\pi]$ with the usual topology from $\mathbb{R}$. Let $Y = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the subspace topology from $\mathbb{R}^2$. Define

$$p : X \rightarrow Y, \quad p(t) = (\cos t, \sin t).$$

Then, p is surjective and continuous. p is not open unfortunately: $[0, a)$ will not be mapped to an open set. But p is closed. Later when we have more tools, this can be proved easily. But now, let's use techniques we have to prove it.

Proof that p is closed. Let $F \subset [0, 2\pi]$ be closed. We show $p(F)$ is closed in S^1 . Since $S^1 \subset \mathbb{R}^2$ is a metric space, it suffices to use sequences.

Take any sequence $(y_n) \subset p(F)$ with $y_n \rightarrow y \in S^1$. For each n choose $t_n \in F$ such that $p(t_n) = y_n$. The sequence (t_n) lies in the bounded interval $[0, 2\pi]$, so by Bolzano–Weierstrass it has a convergent subsequence $t_{n_k} \rightarrow t \in [0, 2\pi]$. Because F is closed and each $t_{n_k} \in F$, we have $t \in F$. By continuity of p ,

$$p(t_{n_k}) \longrightarrow p(t).$$

But $p(t_{n_k}) = y_{n_k} \rightarrow y$, hence $y = p(t) \in p(F)$. Thus every limit of a convergent sequence in $p(F)$ lies in $p(F)$; so $p(F)$ is closed in S^1 .

Therefore p sends closed subsets of $[0, 2\pi]$ to closed subsets of S^1 , i.e. p is a closed map. Since p is surjective and continuous, it is a quotient map. $\square$

In this way, we can identify S^1 with the quotient space obtained by attaching end points of an interval $[0, 2\pi]/(0 \sim 2\pi)$.

Remark 3.108. The gluing maps in the interval, torus, and sphere examples above are quotient maps when the targets are endowed with subspace topologies. In fact, these maps are closed, which we can prove later when we learn about compactness. Right now, we can just use the sequence argument as in the previous example.

3.8.1 Universal properties: products and quotients

Both constructions can be described by how continuous maps factor through them. For a product, we specify a map *into* the product by its coordinates. For a quotient, we obtain a map *out of* the quotient from a map that is constant on each fiber. In each case, the resulting map is unique.

Here *universal* means that the property holds for every topological space Z and every choice of maps satisfying the stated conditions.

Theorem 3.109 (Universal property of the product). *Let $\{X_i\}_{i \in I}$ be a family of topological spaces, and let $P = \prod_{i \in I} X_i$ have the product topology, with coordinate projections $\pi_i : P \rightarrow X_i$. Let Z be any topological space and let $g_i : Z \rightarrow X_i$ be a function for each $i \in I$. Then there is a unique function $g : Z \rightarrow P$ such that*

$$\pi_i \circ g = g_i \quad \text{for every } i \in I.$$

It is defined by $g(z) = (g_i(z))_{i \in I}$. Moreover, g is continuous if and only if g_i is continuous for every $i \in I$.

The following diagram commutes for every $i \in I$:

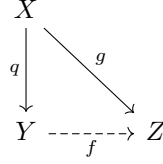
$$\begin{array}{ccc} Z & \xrightarrow{\quad g \quad} & P \\ & \searrow g_i & \downarrow \pi_i \\ & & X_i \end{array}$$

Proof. The formula $g(z) = (g_i(z))_{i \in I}$ defines a function $g : Z \rightarrow P$ with $\pi_i \circ g = g_i$ for every $i \in I$. If $h : Z \rightarrow P$ also satisfies these identities, then $h(z)$ and $g(z)$ have the same coordinates for every $z \in Z$. Hence $h = g$.

If g is continuous, then each $g_i = \pi_i \circ g$ is continuous, since the projections are continuous. Conversely, if every g_i is continuous, then g is continuous by Theorem 3.91. $\square$

Theorem 3.110 (Universal property of a quotient map). *Let $q : X \rightarrow Y$ be a quotient map and let Z be a topological space. Let $g : X \rightarrow Z$ be such that g is constant on each fiber $q^{-1}(\{y\})$. Then there is a unique function $f : Y \rightarrow Z$ such that $f \circ q = g$, defined by $f(q(x)) = g(x)$. Moreover,*

1. f is continuous iff g is continuous.
2. f is a quotient map iff g is a quotient map.



Proof. The formula $f(q(x)) = g(x)$ is well-defined because g is constant on each fiber of q . Since q is surjective, this formula defines f on all of Y and determines it uniquely.

- (1) ($\Rightarrow$) Composition of continuous maps is continuous.
 ($\Leftarrow$) Suppose g is continuous. Let $V \subseteq Z$ be open. Then

$$(f \circ q)^{-1}(V) = q^{-1}(f^{-1}(V))$$

is open in X . Since q is a quotient map, $f^{-1}(V)$ is open in Y ; hence f is continuous.

(2) Since q is surjective, f is surjective if and only if g is; their continuity is equivalent by (1). It remains to check the condition on open sets.

($\Leftarrow$) Suppose g is a quotient map. Let $U \subseteq Z$ be such that $f^{-1}(U)$ is open in Y . Then

$$q^{-1}(f^{-1}(U)) = (f \circ q)^{-1}(U) = g^{-1}(U)$$

is open in X . Since g is a quotient map, U is open in Z . Hence f is a quotient map.

($\Rightarrow$) Suppose f is a quotient map. Let $U \subseteq Z$ be such that $g^{-1}(U)$ is open in X . Then

$$q^{-1}(f^{-1}(U)) = (f \circ q)^{-1}(U) = g^{-1}(U)$$

is open in X . Since q is a quotient map, $f^{-1}(U)$ is open in Y . Since f is a quotient map, U is open in Z . Hence g is a quotient map. The same proof shows that the composition of two quotient maps is a quotient map. $\square$