

3.9 Homotopy

Example 3.111. $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. How about $\mathbb{R}^2$ vs $\mathbb{R}^3$?

We would like to answer this question to some extent. In the process of answering this question, we will introduce a very important concept in topology, called homotopy.

Definition 3.112. Let f and g be continuous maps from a topological space X to a topological space Y . We say that f and g are *homotopic* if there exists a continuous map $H: X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. H is called a *homotopy* between f and g . We write $f \simeq g$ to indicate that f and g are homotopic.

Example 3.113. Any continuous map $f: X \rightarrow Y$ is homotopic to itself via the constant homotopy $H(x, t) = f(x)$.

Example 3.114. In $\mathbb{R}^n$ the identity map $f(x) = x$ is homotopic to the constant map $g(x) = 0$ via the *straight-line homotopy*

$$H(x, t) = (1 - t)x.$$

Example 3.115. More generally, if Y is a convex subset of some $\mathbb{R}^n$ and $f, g: X \rightarrow Y$ are continuous, then

$$H(x, t) = (1 - t)f(x) + tg(x)$$

defines a homotopy $f \simeq g$. Thus any two maps into a convex (in particular, contractible) space are homotopic. The target matters: regarded as maps $S^1 \rightarrow D^2$ the identity and any constant map are homotopic, but as maps $S^1 \rightarrow S^1$ they need not be. This is in fact quite nontrivial and we will return to it later.

Proposition 3.116. *Homotopy is an equivalence relation on $C(X, Y)$, the set of continuous maps $X \rightarrow Y$.*

Proof. Reflexivity: $f \simeq f$ via the constant homotopy $H(x, t) = f(x)$.

Symmetry: if H is a homotopy $f \simeq g$, then $H'(x, t) = H(x, 1 - t)$ is a homotopy $g \simeq f$.

Transitivity: if H_1 is a homotopy $f \simeq g$ and H_2 a homotopy $g \simeq h$, define

$$H(x, t) = \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity follows from the pasting lemma (here we used the fact that the product of two closed sets is closed) because the two pieces agree at $t = \frac{1}{2}$. Then H is a homotopy $f \simeq h$. □

Proposition 3.117 (Composites of homotopic maps are homotopic).

If $f_1, g_1: X \rightarrow Y$ and $f_2, g_2: Y \rightarrow Z$ with $f_1 \simeq g_1$ and $f_2 \simeq g_2$, then $f_2 \circ f_1 \simeq g_2 \circ g_1$.

Proof. Let $H_1: X \times I \rightarrow Y$ be a homotopy $f_1 \simeq g_1$ and $H_2: Y \times I \rightarrow Z$ a homotopy $f_2 \simeq g_2$. Then $f_2 \circ H_1$ is a homotopy $f_2 \circ f_1 \simeq f_2 \circ g_1$. Define $H(x, t) = H_2(g_1(x), t)$; this is continuous as a composite and gives a homotopy $f_2 \circ g_1 \simeq g_2 \circ g_1$. Transitivity yields $f_2 \circ f_1 \simeq g_2 \circ g_1$. $\square$

With the notion of homotopy between functions, we can define homotopy equivalence between spaces.

Definition 3.118. Two spaces X and Y are *homotopy equivalent* if there exist continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. In this case f and g are homotopy inverses and we write $X \simeq Y$.

Remark 3.119. Homotopy equivalence is an equivalence relation on spaces. Any homeomorphism is a homotopy equivalence (take $g = f^{-1}$). The converse fails; see the example below.

One special case of homotopy equivalence is when one of the spaces is a point.

Definition 3.120. A space X is *contractible* if the identity map id_X is homotopic to a constant map $c_{x_0}: X \rightarrow X$ sending every point to x_0 . Equivalently, there exists $H: X \times I \rightarrow X$ with $H(x, 0) = x$ and $H(x, 1) = x_0$ for all x .

Theorem 3.121. A space X is contractible iff it is homotopy equivalent to a one-point space.

Proof. If X is contractible pick $x_0 \in X$ and set $Y = \{x_0\}$. Let $j: Y \hookrightarrow X$ be inclusion and $c: X \rightarrow Y$ the constant map. Then $c \circ j = id_Y$ and $j \circ c = c \simeq id_X$, so $X \simeq Y$.

Conversely, suppose $f: X \rightarrow Y$ and $g: Y \rightarrow X$ exhibit a homotopy equivalence where Y is a point. Then $g \circ f$ is constant and $g \circ f \simeq id_X$, giving a homotopy from id_X to a constant map, so X is contractible. $\square$

Example 3.122. Every convex subset of $\mathbb{R}^n$ (in particular $\mathbb{R}^n$ and any disk D^n) is contractible via the straight-line homotopy $H(x, t) = (1 - t)x + tx_0$.

Example 3.123. $\mathbb{R}^n$ is contractible but $\mathbb{R}^n$ is not homeomorphic to a point.

Example 3.124. S^1 is not contractible. We will prove this later.

The contractible case is the trivial case of homotopy equivalence. More generally, we can often find a smaller subspace which is homotopy equivalent to the whole space. This is made precise by the following definitions.

Definition 3.125. Let $A \subseteq X$. A map $r: X \rightarrow A$ is a *retraction* if $r(a) = a$ for all $a \in A$ (equivalently $r \circ i = id_A$ where $i: A \hookrightarrow X$ is inclusion).

Definition 3.126. A subspace $A \subseteq X$ is a *deformation retract* of X if there is a retraction $r: X \rightarrow A$ such that there is a homotopy $H: X \times I \rightarrow X$ with $H(x, 0) = x$, $H(x, 1) = r(x)$, and $H(a, t) = a$ for all $a \in A$, $t \in I$. We call H a *deformation retraction*.

Proposition 3.127. *If A is a deformation retract of X then $A \simeq X$. Thus a deformation retract gives a special homotopy equivalence (with inverse given by the retraction).*

Proof. Let $r: X \rightarrow A$ be a retraction and $i: A \hookrightarrow X$ inclusion with $i \circ r \simeq \text{id}_X$. Then $r \circ i = \text{id}_A$, so i and r are homotopy inverses. $\square$

Example 3.128. 1. The circle S^1 is a deformation retract of any annulus $A = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq R\}$ with the retraction $r(x) = x/\|x\|$ and the homotopy $H(x, t) = (1 - t)x + tr(x)$.

2. $\mathbb{R}^n \setminus \{0\}$ deformation retracts onto S^{n-1} : the retraction is $r(x) = x/\|x\|$ and the homotopy is $H(x, t) = (1 - t)x + tr(x)$.

Example 3.129. Not every retraction is a deformation retraction. In fact, for any X and $x_0 \in X$, the constant map $c_{x_0}: X \rightarrow \{x_0\}$ is a retraction if $\{x_0\}$ is a retract of X , but it is a deformation retraction only if X is contractible.

Theorem 3.130. *There is no retraction $D^2 \rightarrow S^1$.*

This can be proved using algebraic topology (fundamental group) or using homology. We will give a proof without these later.

Now we are ready to answer the question whether $\mathbb{R}^2$ and $\mathbb{R}^3$ are homeomorphic.

Theorem 3.131. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.*

Proof. Suppose on the contrary there is a homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Then f restricts to a homeomorphism $\mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{f(0)\}$. But $\mathbb{R}^2 \setminus \{0\} \simeq S^1$ and $\mathbb{R}^3 \setminus \{f(0)\} \simeq S^2$. So $S^1 \simeq S^2$. We will show this is not true.

We will use the following lemmas.

Lemma 3.132. *The identity map on S^1 is not homotopic to a constant map. (This proves S^1 is not contractible.)*

Lemma 3.133. *Every map $S^1 \rightarrow S^2$ is homotopic to a constant map.*

Now suppose $S^1 \simeq S^2$ via maps $f: S^1 \rightarrow S^2$ and $g: S^2 \rightarrow S^1$. Then, we know that f is homotopic to a constant map $c: S^1 \rightarrow S^2$. So $g \circ f: S^1 \rightarrow S^1$ is homotopic to $g \circ c$, which is a constant map. But $g \circ f \simeq \text{id}_{S^1}$, which contradicts the first lemma. $\square$

The proof of the first lemma is as follows.

Proof of Lemma 3.132. We need the following lemma.

Lemma 3.134 (Lifting lemma). *Let $p: \mathbb{R} \rightarrow S^1$ be $p(t) = e^{it}$. If $f: S^1 \rightarrow S^1$ is homotopic to a constant map, then f lifts, i.e., there exists a continuous $\tilde{f}: S^1 \rightarrow \mathbb{R}$ with $p \circ \tilde{f} = f$.*

Proof. We first show that if $g : S^1 \rightarrow S^1$ lifts and $h : S^1 \rightarrow S^1$ satisfies $h(x) \neq -g(x)$ for all $x \in S^1$, then h lifts. Indeed, $h(x)/g(x) \in S^1$ (viewed as complex numbers) is never -1 , so we can write $h(x)/g(x) = e^{i\psi(x)}$ with a unique continuous $\psi(x) \in (-\pi, \pi)$. If $\tilde{g}$ is a lift of g , i.e., $g(x) = e^{i\tilde{g}(x)}$, then $h(x) = g(x)e^{i\psi(x)} = e^{i(\tilde{g}(x) + \psi(x))}$, so $\tilde{h}(x) = \tilde{g}(x) + \psi(x)$ is a lift of h .

Now, let $H : S^1 \times I \rightarrow S^1$ be a homotopy from the constant map $c(z) = 1$ to f . Equip S^1 with the chord metric $d(u, v) = |u - v|$ (so $d(u, v) = 2$ iff $v = -u$). Since H is uniformly continuous on the compact set $S^1 \times I \subset \mathbb{R}^3$, choose $\delta > 0$ so that

$$|t - s| < \delta \implies d(H(z, t), H(z, s)) < 2 \quad \text{for all } z \in S^1.$$

Pick $0 = t_0 < t_1 < \dots < t_N = 1$ with $t_{j+1} - t_j < \delta$.

The map $H(\cdot, 0) = c_1$ lifts to $\tilde{H}_N : S^1 \rightarrow \mathbb{R}$ by $\tilde{H}_N(z) \equiv 0$.

Inductive step. Suppose H_j lifts. For any z , the points $H(z, t_j) \neq H(z, t_{j+1})$ since $d(H(z, t_j), H(z, t_{j+1})) < 2$, so $H(z, t_{j+1})$ lifts as well.

By induction we have that $f = H(\cdot, 1)$ lifts. $\square$

Now, suppose id_{S^1} is homotopic to a constant map. Then, by the lifting lemma, id_{S^1} lifts to a continuous map $\tilde{f} : S^1 \rightarrow \mathbb{R}$ with $p \circ \tilde{f} = \text{id}_{S^1}$. The lift $\tilde{f}$ must be injective because id_{S^1} is injective. This would violate the intermediate value theorem: choose $\alpha \in (\phi(-1), \phi(1))$ (here we assume $\phi(-1) < \phi(1)$). Then both arcs in S^1 from -1 to 1 must contain a point z with $\phi(z) = \alpha$, contradicting injectivity. $\square$

Proof of Lemma 3.133. Let $f : S^1 \rightarrow S^2$ be continuous. Pick $p \in S^2 \setminus f(S^1)$. (f could be surjective but let's ignore this case and it can be fixed using some more arguments).

Observe that $S^2 - \{p\} \cong \mathbb{R}^2$. Let $g : S^2 - \{p\} \rightarrow \mathbb{R}^2$ be a homeomorphism. Then, $g \circ f : S^1 \rightarrow \mathbb{R}^2$ is continuous. Since $\mathbb{R}^2$ is convex, $g \circ f$ is homotopic to a constant map via the straight-line homotopy. Composing with g^{-1} gives a homotopy from f to a constant map. More precisely, if $H_1 : S^1 \times I \rightarrow \mathbb{R}^2$ is a homotopy from $g \circ f$ to a constant map, then $H_2 = g^{-1} \circ H_1 : S^1 \times I \rightarrow S^2 - \{p\} \hookrightarrow S^2$ is a homotopy from f to a constant map. $\square$

Brouwer Fixed Point Theorem

Now, we introduce some interesting results and prove some special cases using the fact that S^1 is not contractible.

Theorem 3.135 (No retraction). *There is no retraction $D^n \rightarrow S^{n-1}$ for $n \geq 1$.*

Proof. We prove the case when $n = 2$. Suppose there is a retraction $r : D^2 \rightarrow S^1$. Then, we define $H : S^1 \times I \rightarrow S^1$ as follows:

$$H(x, t) = r(tx).$$

Then, $H(x, 0) = r(0)$ is a constant map and $H(x, 1) = r(x) = x$. So H is a homotopy from the identity map on S^1 to a constant map. This contradicts Lemma 3.132. $\square$

Proof. If $r : D^n \rightarrow S^{n-1}$ were a retraction, compose any map $S^{n-1} \rightarrow D^n$ extending $\text{id}_{S^{n-1}}$ with r to obtain a fixed-point-free self map of D^n , contradicting Brouwer. $\square$

Corollary 3.136 (Brouwer Fixed Point). *For every $n \geq 1$, any continuous map $F : D^n \rightarrow D^n$ has a fixed point, i.e. there exists $x \in D^n$ with $F(x) = x$.*

Proof. Assume F has no fixed point. For each $x \in D^n$ draw the ray starting at $F(x)$ through x and let $r(x)$ be the first intersection of that ray with $S^{n-1} = \partial D^n$. This gives a continuous map $r : D^n \rightarrow S^{n-1}$ which restricts to the identity on S^{n-1} , hence a retraction. But S^{n-1} is not a retract (indeed not a deformation retract) of D^n (proved earlier using homotopy ideas). Contradiction. $\square$

Homotopy equivalence = embedding + retraction

If there is one thing that you need to remember, that is homotopy equivalence is basically embedding + retraction.

First of all, we introduce the notion of embeddings.

Definition 3.137 (Embedding). A continuous map $f : X \rightarrow Y$ is an *embedding* if it is a homeomorphism onto its image $f(X)$ (with the subspace topology inherited from Y). In particular, f is injective.

Definition 3.138. Let X and Y be two disjoint topological spaces. Then, the *disjoint union* of X and Y is the topological space

$$X \sqcup Y$$

whose underlying set is the disjoint union of sets X and Y and whose topology is given by

$$U \subseteq X \sqcup Y \text{ is open} \iff U \cap X \text{ is open in } X \text{ and } U \cap Y \text{ is open in } Y.$$

Equivalently, the topology is given by

$$\tau_{X \sqcup Y} = \{U \sqcup V \mid U \in \tau_X, V \in \tau_Y\}.$$

Definition 3.139 (Mapping cylinder). Given a continuous map $f : X \rightarrow Y$, the *mapping cylinder* of f is

$$M_f = (X \times [0, 1]) \sqcup Y / \sim$$

where $(x, 1) \sim f(x)$ for all $x \in X$. Write $[x, t]$ for the class of (x, t) and $[y]$ for $y \in Y$. There are canonical embeddings

$$i_X : X \hookrightarrow M_f, \quad x \mapsto [x, 0], \quad \text{and} \quad i_Y : Y \hookrightarrow M_f, \quad y \mapsto [y].$$

There is a deformation retraction $H : M_f \times I \rightarrow M_f$ onto $i_Y(Y)$ given by $H([x, t], s) = [x, (1-s)t + s]$ and $H([y], s) = [y]$, so M_f deformation retracts onto Y .

Lemma 3.140. *For any continuous map $f : X \rightarrow Y$, the map $H : M_f \times I \rightarrow M_f$ defined by $H([x, t], s) := [x, (1-s)t + s]$ and $H([y], s) := [y]$ is a deformation retraction of M_f onto $i_Y(Y)$.*

Lemma 3.141. *$f : X \rightarrow Y$ is a homotopy equivalence if and only if $i_X(X)$ is a deformation retract of M_f .*

Theorem 3.142 (Common deformation–retract envelope).

For topological spaces X and Y , the following are equivalent:

1. *X and Y are homotopy equivalent.*
2. *There exists a space Z and embeddings $i_X : X \hookrightarrow Z$, $i_Y : Y \hookrightarrow Z$ such that $i_X(X)$ and $i_Y(Y)$ are each deformation retracts of Z (possibly via two different homotopies).*

Proof. (2) $\Rightarrow$ (1): If Z deformation retracts onto $i_X(X)$ and onto $i_Y(Y)$, then $Z \simeq X$ and $Z \simeq Y$, hence $X \simeq Y$ by transitivity.

(1) $\Rightarrow$ (2): Suppose $f : X \rightarrow Y$ is a homotopy equivalence. Form the mapping cylinder M_f , which admits deformation retractions onto Y and X . $\square$

4 Continuity and Convergence

4.1 Convergence in Topological Spaces

We will try to answer the following questions:

1. Can we define sequential convergence in general topological spaces?
2. If so, does this convergence describe the notion of topology?

In metric space setting, we have the notion of convergence of a sequence (in the language of open balls):

Proposition 4.1. *A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space X converges to a point $x \in X$ if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$.*

This motivates the following definition in general topological spaces:

Definition 4.2. A sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space X is said to converge to a point $x \in X$ if for every neighbourhood U of x , there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Problem 4.3. A sequence (x_n) in a metric space converges in the metric sense if and only if it converges in the topological sense defined above.

Proof. ($\Rightarrow$) Let (x_n) be a sequence in a metric space (X, d) converging to $x \in X$ in the metric sense. For every neighborhood U of x , there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Then, there exists N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In particular, $x_n \in B_\epsilon(x) \subset U$ for all $n \geq N$. So we have $x_n \rightarrow x$ in the topological sense.

($\Leftarrow$) Let (x_n) be a sequence in a metric space (X, d) converging to $x \in X$ in the topological sense. For every $\epsilon > 0$, the open ball $B_\epsilon(x)$ is a neighbourhood of x , so there exists N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$. This means that $d(x_n, x) < \epsilon$ for all $n \geq N$. Hence, (x_n) converges to x in the metric sense. $\square$

Proposition 4.4. *Let X be a topological space. For $A \subseteq X$ and $x \in X$, if there is a sequence $(a_n) \subseteq A$ with $a_n \rightarrow x$, then $x \in \overline{A}$.*

Proof. Let U be any neighbourhood of x . Then there exists N such that $a_n \in U$ for all $n \geq N$. In particular, $a_N \in A \cap U$, so $A \cap U \neq \emptyset$. Since U was arbitrary, we have $x \in \overline{A}$. $\square$

We know the converse is true in metric spaces. Is it true in general topological spaces?

Example 4.5. Let $X = \mathbb{R}$ be endowed with the *countable complement topology*: a set $U \subseteq X$ is open iff $X \setminus U$ is countable or $U = \emptyset$.

- Any sequence (x_n) converging to x is eventually equal to x . Indeed, we let $B = \{x_n : x_n \neq x\}$ and hence $U = X \setminus B$ is an open neighborhood of x . Then, there exists N such that $x_n \in U$ for all $n \geq N$, i.e., $x_n \notin B$ for all $n \geq N$. This means that $x_n = x$ for all $n \geq N$.
- The closed sets are exactly the countable sets and X itself. Hence, if $A \subsetneq X$ is uncountable, then $\overline{A} = X$.

Thus $\overline{[0, 1]} = \mathbb{R}$, but there is no sequence in $[0, 1]$ converging to 2.