1 Set theory

In this section we review/develop the set-theoretic background for the course.

1.1 Basic definitions

Naive definition: A set is an aggregate of things, called *elements* of the set. We usually use capital letters to denote sets, such as A, B, C, and lowercase letters for elements, such as a, b, c. The notation $a \in A$ means that a is an element of the set A. We write $a \notin A$ to mean that a is not an element of the set A.

Example 1.1. Some basic examples of sets include:

- The natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$
- The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The real numbers $\mathbb{R}$

Set calculus A set A is a subset of a set B if every element of A is also an element of B. We write $A \subseteq B$ to denote that A is a subset of B. If A is a subset of B but not equal to B, we say that A is a *proper subset* of B and denote it by $A \subsetneq B$.

We let $\mathcal{P}(A)$ denote the *power set* of A, which is the set of all subsets of A.

We say two sets A and B are *equal* if they have the same elements, that is, if $A \subseteq B$ and $B \subseteq A$. We write $A = B$ to denote that A and B are equal.

We let $A \backslash B$ denote the *set difference* of A and B, which is the set of elements in A that are not in B. Formally, $A \backslash B = \{x \in A : x \notin B\}$. When $B \subseteq A$, we say $A \backslash B$ is the *complement* of B in A.

The empty set, denoted $\varnothing$, is a set that contains no elements. It is a subset of every set.

Definition 1.2. A family of sets $\mathcal{A}$ is a set whose elements are sets. Usually, we can write the family of sets as

$$\mathcal{A} = \{A_i\}_{i \in I}$$

for some index set I.

If $\mathcal{A}$ is a family of sets, we define the union of the family as

$$\bigcup_{A \in \mathcal{A}} A := \{x | \exists A \in \mathcal{A}, x \in A\}.$$

and the intersection as

$$\bigcap_{A \in \mathcal{A}} A := \{x | \forall A \in \mathcal{A}, x \in A\}.$$

Theorem 1.3. *Let X be a set and $\{A_i\}_{i\in I} \subseteq \mathcal{P}(X)$. Then (De Morgan's laws)*

$$X \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I}(X \setminus A_i), \qquad X \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I}(X \setminus A_i).$$

Moreover, for any $A \subseteq X$ and $\{B_i\}_{i\in I} \subseteq \mathcal{P}(X)$ (distributive laws),

$$A \cap \left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I}(A \cap B_i), \qquad A \cup \left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I}(A \cup B_i).$$

Example 1.4 (Russell's paradox)**.** Let P denote the set of all sets. Then, form $Q := \{A \in P | A \notin A\}$. Is $Q \in Q$?

More axiomatic treatment of sets can be found in Zermelo-Fraenkel set theory (ZF), which provides a formal foundation for set theory and avoids paradoxes like Russell's paradox by carefully restricting the kinds of sets that can be formed. For example, under ZF axioms, the collection of all sets is not a set but a *proper class*, because it violates one of the axioms requiring that a set cannot contain itself as an element.

1.2 Functions

Cartesian product The *Cartesian product* of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Formally, an ordered pair can be defined as follows and define the cartesian product as a set of sets:

$$(a, b) := \{\{a\}, \{a, b\}\}.$$

Remark 1.5. If one set is empty, then the Cartesian product is also empty:

$$A \times \varnothing = \varnothing \times B = \varnothing.$$

Definition 1.6. A function $f\colon A \to B$ is a subset of the Cartesian product $A \times B$ such that for any $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.

Example 1.7. $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2 + 1$ is a function $\{(x, x^2 + 1) \in \mathbb{R} \times \mathbb{R}\}$.

Formally, $g : \mathbb{R} \to \mathbb{R}_+$ defined by $x \mapsto x^2 + 1$ is a different function where $\mathbb{R}_+ := \{x \in \mathbb{R} | x > 0\}$.

Of course, since each $a \in A$ is associated with a unique $b \in B$, we often write $f(a) = b$ and then

$$f = \{(a, f(a)) : a \in A\} \subseteq A \times B.$$

For any $C \subseteq A$, we define the image of C under f as the set

$$f(C) = \{f(a) : a \in C\}.$$

We call $f(A)$ the *image* of f.

For any $D \subseteq B$, we define the preimage of D under f as the set

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$

If the image of f is equal to B, we say that f is *onto* or *surjective*. If for any $a_1 \neq a_2$ we have that $f(a_1) \neq f(a_2)$, we say that f is *one-to-one* or *injective*. If a function is both injective and surjective, it is called a *bijection*. We usually denote a bijection by $f : A \cong B$.

Let $f\colon A \to B$ and $g\colon B \to C$ be two functions. The *composition* of f and g is the function $g \circ f\colon A \to C$ defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

Example 1.8. For any nonempty set A, we define the identity function $id_A : A \to A$ by $id_A(a) = a$ for all $a \in A$. It is clear that id_A is a bijection. Moreover, for any function $f : A \to B$, we have $f \circ id_A = f$ and $id_B \circ f = f$.

Cartesian products of families of sets If $\{A_i\}_{i \in I}$ is a family of sets indexed by a set I, the Cartesian product of the family is defined as

$$\prod_{i \in I} A_i = \{f : I \to \bigcup_{i \in I} A_i : f(i) \in A_i \text{ for all } i \in I\}.$$

Example 1.9. If $I = \{1, 2, \ldots, n\}$ is a finite set, then $\prod_{i \in I} A_i$ can be identified with the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for each i.

1.3 Relations

Definition 1.10. A *relation* on a set S is any subset $R \subseteq S \times S$. If $(a, b) \in R$, we say that a is related to b and write aRb.

A relation R is called

- *reflexive* if aRa for all $a \in S$;
- *symmetric* if aRb implies bRa;
- *transitive* if aRb and bRc imply aRc;
- *antisymmetric* if aRb and bRa imply $a = b$.

1.3.1 Equivalence relations

Definition 1.11. We say that R is an *equivalence relation* if it is reflexive, symmetric, and transitive. An equivalence relation is usually denoted by $\sim$ and if $(a, b) \in \sim$ we write $a \sim b$.

Example 1.12. The relation R on $\mathbb{Z}$ defined by aRb if and only if $a - b$ is even is an equivalence relation.

Example 1.13. Let $f : A \to B$ be any function. The relation R on A defined by aRb if and only if $f(a) = f(b)$ is an equivalence relation.

Example 1.14. Let T denote the set of all triangles in the plane. The relation R on T defined by $\triangle_1 R \triangle_2$ if and only if $\triangle_1$ is similar to $\triangle_2$ is an equivalence relation.

Definition 1.15. Let A be a non empty set. For any $a \in A$, we define the equivalence class of a as

$$[a] = \{\, b \in A : a \sim b \,\}.$$

Example 1.16. Consider the equivalence relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ if and only if $a - b$ is even. Then, there are two equivalence classes: the even integers and the odd integers. Specifically,

$$[0] = \{\ldots -4, -2, 0, 2, 4, 6, \ldots\}, \qquad [1] = \{\ldots -5, -3, -1, 1, 3, 5, \ldots\}.$$

Definition 1.17. Let A be a non empty set. A partition of A is a family of nonempty subsets $\{A_i\}_{i \in I}$ such that:

- $A = \bigcup_{i \in I} A_i$
- $A_i \cap A_j = \varnothing$ for all $i \neq j$

We sometimes will call each A_i a *block* of the partition.

Example 1.18. Consider the set $A = \{1, 2, 3, 4, 5\}$. Then, $\{\{1, 2\}, \{3\}, \{4, 5\}\}$ is a partition of A.

Example 1.19. For any set A, we call $\{\{a\}\}_{a \in A}$ the *discrete partition* of A and we call $\{A\}$ the *trivial partition* of A.

We denote by $\mathrm{Part}(A)$ the set of all partitions of A and by $\mathrm{EqRel}(A)$ the set of all equivalence relations on A.

Theorem 1.20 (Partitions and equivalence relations)**.** *Let A be a nonempty set. There is a one-to-one correspondence between partitions of A and equivalence relations on A:*

- *Define $f : \mathrm{Part}(A) \to \mathrm{EqRel}(A)$ by sending a partition $\{A_i\}_{i \in I}$ to the equivalence relation $\sim$ on A defined by $a \sim b$ if and only if $\exists i \in I$ with $a, b \in A_i$.*
- *Conversely, define $g : \mathrm{EqRel}(A) \to \mathrm{Part}(A)$ by sending an equivalence relation $\sim$ on A to the family of equivalence classes*

  $$\{[a] : a \in A\}$$

  which is a partition of A.

Then, f and g are bijections and they are inverse to each other, i.e.,

$$g \circ f = \mathrm{id}_{\mathrm{Part}(A)}, \qquad f \circ g = \mathrm{id}_{\mathrm{EqRel}(A)}.$$

1.3.2 Partial orders

Definition 1.21. A relation R on A is a *partial order* if it is reflexive, antisymmetric, and transitive. A partial order is usually denoted by $\leq$. A set equipped with a partial order is called a *partially ordered set* or *poset*, and is denoted by $(A, \leq)$. When $a \leq b$ and $a \neq b$, we write $a < b$.

Example 1.22. Consider the set of real numbers $\mathbb{R}$ with the usual order relation $\leq$. This is a partial order because it is reflexive (every number is less than or equal to itself), antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$), and transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$).

Just a funny note: we humans know that $0.9 > 0.11$ but somehow many LLMs think $0.9 < 0.11$. So be careful when using AIs in your studies :)

Example 1.23. Let A be a non empty set and consider its power set $\mathcal{P}(A)$, the set of all subsets of A. Define, for $B, C \in \mathcal{P}(A)$,

$$B \leq C \quad \Longleftrightarrow \quad B \subseteq C.$$

Then $(\mathcal{P}(A), \leq)$ is a partial order.

Example 1.24. Partitions of a fixed set A are partially ordered by refinement: for partitions $\pi, \sigma \in \mathrm{Part}(A)$

$$\pi \leq \sigma \quad \text{if and only if} \quad (\forall B \in \pi)(\exists C \in \sigma)\ B \subseteq C.$$

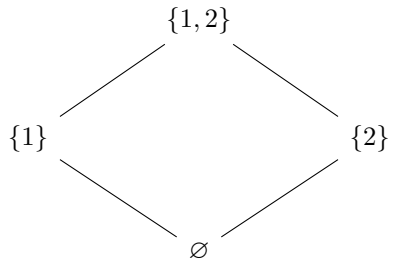
Thus $\pi \leq \sigma$ means "π is finer than σ" (equivalently, "σ is coarser than π").

Example 1.25 (Refinement — simple illustration)**.** Let $A = \{1, 2, 3\}$. Consider the partitions

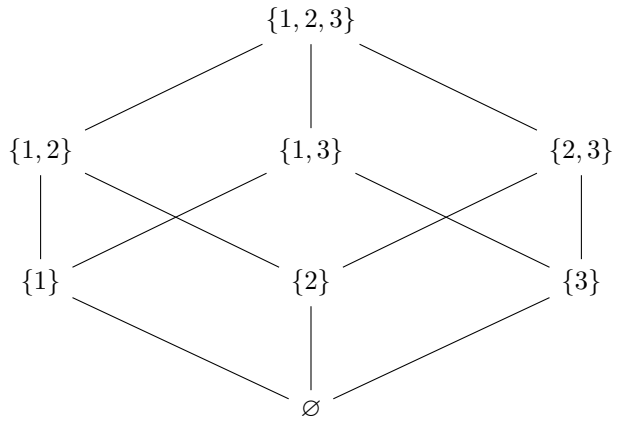
$$\pi = \{\{1\}, \{2\}, \{3\}\} \qquad \text{(the discrete partition)}, \qquad \sigma = \{\{1, 2\}, \{3\}\}.$$

We have $\pi \leq \sigma$ because each block of π is contained in a block of σ: $\{1\} \subseteq \{1, 2\},\ \{2\} \subseteq \{1, 2\},\ \{3\} \subseteq \{3\}$.

Definition 1.26. *Hasse diagram.* Let $(A, \leq)$ be a finite poset. For $x, y \in A$, say that y covers x if $x < y$ and there is no $z \in A$ with $x < z < y$. The Hasse diagram of $(A, \leq)$ is the directed acyclic graph with vertex set A and an edge $x \to y$ precisely when x is covered by y. By convention it is drawn with y placed higher than x and arrowheads omitted, and transitive edges are suppressed.



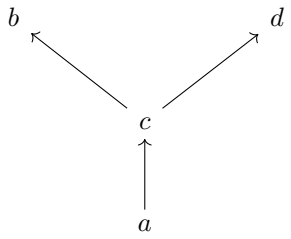
Example 1.27. Some pairs of elements in a poset may not be comparable, i.e., there may be $a, b \in A$ such that neither $a \leq b$ nor $b \leq a$.



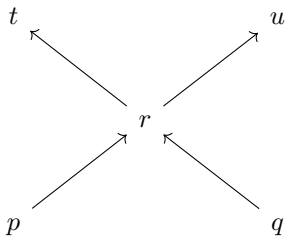
Definition 1.28. Let $(A, \leq)$ be a poset. A *maximal element* of A is an element $m \in A$ such that there is no $x \in A$ with $m < x$. A *minimal element* of A is an element $n \in A$ such that there is no $x \in A$ with $x < n$.

The *largest element* of A, if it exists, is an element $u \in A$ such that $u \geq x$ for all $x \in A$. The *smallest element* of A, if it exists, is an element $l \in A$ such that $l \leq x$ for all $x \in A$.

Example 1.29. Consider the poset below. Indicate minimal/maximal and smallest/largest elements.



Example 1.30. Consider the poset below. Indicate minimal/maximal and smallest/largest elements.



There is one case when the notion of maximal elements coincides with that of largest elements: when the poset is totally ordered.

Definition 1.31 (Linear (total) order)**.** A partial order $\leq$ on a set A is a linear (or total) order if for all $a, b \in A$ one has $a \leq b$ or $b \leq a$ (comparability).

Remark 1.32. A minimal element in a linear poset is also the smallest element. Similarly, a maximal element is also the largest element.

Definition 1.33 (Well-order)**.** A linear poset $(A, \leq)$ is well-ordered if every nonempty subset $B \subseteq A$ has a minimal element.

Example 1.34.
- The set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the usual order $\leq$ is well-ordered. Be careful here and ask yourself why. This can be either taken as an axiom in Peano Axioms or proved under ZF.
- The set of real numbers $\mathbb{R}$ with the usual order $\leq$ is not well-ordered. $[0, 1]$ is not well-ordered as well.

Example 1.35.
- We define the lexicographic order on $\mathbb{N} \times \mathbb{N}$ as follows: for $(m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}$, we say that $(m_1, n_1) \leq (m_2, n_2)$ if either $m_1 < m_2$ or $m_1 = m_2$ and $n_1 \leq n_2$. Then, $(\mathbb{N} \times \mathbb{N}, \leq)$ is a well-ordered set.
- We can similarly define the lexicographic order on $\mathbb{N}^k$ for any finite $k \in \mathbb{N}$, which makes $(\mathbb{N}^k, \leq)$ a well-ordered set as well.
- However, the lexicographic order on $\mathbb{N}^{\mathbb{N}}$ is not well-ordered: consider the subset $B = \{(2, 1, 1, \ldots) > (1, 2, 1, \ldots) > \ldots\}$. Then, B has no minimal element.

The lexicographic order on $\mathbb{N}^{\mathbb{N}}$ is not well-ordered. In fact, no one has found an explicit well-order on $\mathbb{N}^{\mathbb{N}}$ so far. However, it turns out that in theory, one can find a partial order on $\mathbb{N}^{\mathbb{N}}$ so that it is well-ordered.

Theorem 1.36 (Well-ordering theorem)**.** *Every set can be well-ordered.*

There is some caveat of this theorem though. We will mention a bit more later on.

Definition 1.37. Let A be a well-ordered poset. Then, A has a smallest element which we denote by 0_A. For any $a, b \in A$, we define the interval $[a, b]$ as

$$[a, b] = \{x \in A : a \leq x \leq b\}.$$

We can similarly define $[a, b)$, $(a, b]$ and (a, b).

Definition 1.38. Let $(A, \leq)$ be a well-ordered set. A subset $S \subset A$ is called an *initial segment* if for any $x \in S$ and $y \in A$, $y < x$ implies $y \in S$.

Example 1.39. $[0_A, a)$ and $[0_A, a]$ are both initial segments of A. Any nonempty initial segment S of A can be written as $\bigcup_{s \in S} [0_A, s]$.

Order preserving maps An order preserving map between two posets $(A, \leq_A)$ and $(B, \leq_B)$ is a function $f\colon A \to B$ such that $x \leq_A y$ implies $f(x) \leq_B f(y)$. An *order isomorphism* is a bijective order preserving map $f : A \to B$ such that $x \leq_A y \iff f(x) \leq_B f(y)$.

Example 1.40. $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto ax + b$ is an order isomorphism if $a > 0$.

There are infinitely many order isomorphisms $f : \mathbb{R} \to \mathbb{R}$. But if we require the poset to be well-ordered, then the situation changes.

Lemma 1.41. *Let A, B be well-ordered sets. If $f, g : A \to B$ are order isomorphisms, then $f = g$.*

Proof. By the lemma above, we have that $f(0_A) = g(0_A)$. Let $S := \{a \in A : f(a) \neq g(a)\}$ and assume $S \neq \emptyset$. Since A is well-ordered, S has a smallest element, say $0_A \neq b \in S$. Then, we have $f(b) \neq g(b)$ and for all $a \in [0_A, b)$, $f(a) = g(a)$, i.e., $f([0_A, b)) = g([0_A, b))$.

But

$$f([0_A, b)) = [0_B, f(b)) \quad \text{and} \quad g([0_A, b)) = [0_B, g(b)).$$

Thus, we get

$$f(b) = g(b)$$

Contradiction! So we have $f = g$. □

This can be strengthened as follows (this will be your homework)

Lemma 1.42. *Let A, B be well-ordered sets. If f, g are order isomorphisms from A onto initial segments of B, then $f = g$.*

Now, we see an interesting result below saying that one can always somehow "compare" two well-ordered sets.

Theorem 1.43. *Let A, B be two non empty well-ordered sets. Then, there exists an order isomorphism from one of the sets to an initial segment of the other.*

Proof. Let

$$\alpha := \{a \in A : \exists \text{ order-isomorphism } f_a : [0_A, a] \to \text{ an initial segment of } B\}.$$

If $\alpha = A$ then for each $a \in A$ choose such an f_a. By the uniqueness lemma two maps f_{a_1}, f_{a_2} agree on the overlap $[0_A, \min\{a_1, a_2\}]$, so the family $\{f_a\}_{a \in A}$ glues to a single order-isomorphism

$$f : A \to \bigcup_{a \in A} f_a([0_A, a]) = \bigcup_{a \in A} [0_B, f_a(a)],$$

defined by

$$f(x) = \begin{cases} f_a(x) & \text{if } x \in [0_A, a] \text{ for some } a \in A \\ 0_B & \text{if } x = 0_A \end{cases}$$

and the union on the right is an initial segment of B. Thus A is order-isomorphic to an initial segment of B.

If $\alpha \neq A$ let a_0 be the smallest element of $A \setminus \alpha$. For every $a < a_0$ pick an isomorphism $f_a : [0_A, a] \to S_a$ where S_a is an initial segment of B; by uniqueness these glue to an order-isomorphism

$$f : [0_A, a_0) \to \beta := \cup_{a<a_0} S_a$$

where β is an initial segment of B. If $\beta \neq B$ let $b_0 := \min(B \setminus \beta)$. Then $f \cup \{(a_0, b_0)\}$ is an order-isomorphism $[0_A, a_0] \to \beta \cup \{b_0\}$, an initial segment of B, contradicting $a_0 \notin \alpha$. Hence $\beta = B$, so B is order-isomorphic to the initial segment $[0_A, a_0)$ of A.

In either case one of the two sets is order-isomorphic to an initial segment of the other, as claimed. □

1.4 Axiom of Choice

The most commonly used axiomatic system for mathematics is the Zermelo-Fraenkel set theory (ZF).

- ZFC with the Axiom of Choice
- ZF without the Axiom of Choice

Let's see what is this Axiom of Choice.

(Axiom of Choice) If $\{A_i\}_{i\in I}$ is a family of nonempty pairwise disjoint sets, there is a set $B \subset \cup A_i$ such that $B \cap A_i$ is exactly one element for all $i \in I$.

What if the sets are not disjoint?

Definition 1.44. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of nonempty sets. A choice function for $\mathcal{A}$ is a function $f : \mathcal{A} \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

Proposition 1.45. *For any collection of nonempty sets, there exists a choice function.*

Proof. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of nonempty sets. For each $i \in I$ set

$$B_i := \{(i, a) : a \in A_i\}.$$

Then the family $\{B_i\}_{i\in I}$ consists of nonempty pairwise disjoint sets. By the Axiom of Choice (the pairwise disjoint formulation) there exists a set $C \subseteq \bigcup_{i\in I} B_i$ such that $C \cap B_i$ contains exactly one element for every $i \in I$. For each i write the unique element of $C \cap B_i$ as (i, c_i) with $c_i \in A_i$, and define $f : \mathcal{A} \to \bigcup_{i\in I} A_i$ by $f(A_i) = c_i$. Equivalently, view f as a function $f : I \to \bigcup_{i\in I} A_i$ with $f(i) = c_i$. In either view $f(i) \in A_i$ for all i, so f is a choice function for $\mathcal{A}$. □

The reason AC is debatable is that this doesn't constructively provide a way to choose elements from each set. If you have a finite collection of sets, you can simply list the elements and choose one from each set. This idea may still apply when you have a countable collection of sets. But when you have an uncountable collection of sets, the Axiom of Choice becomes more controversial.

1.4.1 Maximum principle and Zorn's lemma

Definition 1.46. A chain is a totally (linearly) ordered subset of a poset.

Definition 1.47. Let A be a poset and $B \subseteq A$. Then, an upper bound of B in A is an element $u \in A$ such that $b \leq u$ for all $b \in B$.

Example 1.48. Consider the set $A = \{1, 2, 3\}$ and the subset $\{\emptyset, \{2\}, \{1, 2, 3\}\} \subset P(A)$ is a chain.

(Maximum Principle) In a poset, every chain is contained in a maximal chain (i.e., a chain that is not properly contained in any other chain).

(Zorn's Lemma) Let A be a poset. If every chain in A has an upper bound in A, then A has a maximal element.

Proposition 1.49 (Maximum principle implies Zorn's lemma)**.** *If we assume the Maximal Principle is true for all posets, then Zorn's Lemma is true for all posets.*

Proof. Let A be a poset. Let $B \subset A$ be a maximal chain. Let u be an upper bound of B in A. If $u < v$, then $B \cup \{v\}$ is a chain containing B, contradicting the maximality of B. Thus, u is a maximal element of A. □

1.4.2 Four equivalent statements

Theorem 1.50. *The following statements are equivalent:*

1. *(Axiom of Choice) If $\{A_i\}_{i \in I}$ is a family of nonempty pairwise disjoint sets, there is a set $B \subset \cup A_i$ such that $B \cap A_i$ is exactly one element for all $i \in I$.*
2. *(Zorn's Lemma) If each chain (linearly ordered set) in a nonempty poset A has an upper bound, then A has a maximal element.*
3. *(Maximal Principle) In a poset, every chain is contained in a maximal chain.*
4. *(Zermelo's Well-Ordering Theorem) Every set can be well-ordered.*

Proof. We've already seen how 3 implies 2. Let's see one more.

$4 \Rightarrow 1$ Make $U := \bigcup A_i$ well ordered. Then, pick $a_i := \min A_i$ since $A_i \subset U$. Then, $B = \{a_i\}_{i \in I}$ is a set such that $B \cap A_i$ is exactly one element for all $i \in I$. Thus, the Axiom of Choice holds.

$2 \Rightarrow 1$. Assume Zorn's Lemma. Let $\{A_i\}_{i \in I}$ be a family of nonempty (pairwise disjoint) sets. Consider the set

$$\mathcal{F} = \{f \mid f \text{ is a function with domain } J \subseteq I,\ f(j) \in A_j\ \forall j \in J\},$$

partially ordered by extension ($f \leq g$ iff $f \subseteq g$). If $\mathcal{C} \subseteq \mathcal{F}$ is a chain, then $g := \bigcup_{f \in \mathcal{C}} f$ is an upper bound of $\mathcal{C}$ (the union is a function since members of the chain agree on overlaps). By Zorn's Lemma there exists a maximal element $f^* \in \mathcal{F}$. If $\operatorname{dom} f^* \neq I$ pick $i_0 \in I \setminus \operatorname{dom} f^*$ and choose $a \in A_{i_0}$ (nonemptiness). Then $f^* \cup \{(i_0, a)\} \in \mathcal{F}$ strictly extends f^*, contradicting maximality. Hence $\operatorname{dom} f^* = I$, so f^* is a choice function. □