Theorem 5.13. *Subspaces and countable products of first (resp. second) countable spaces are first (resp. second) countable.*

Proof. We only prove the first countable case as the second countable case is similar.

Let X be a first countable space and $A \subseteq X$. For each $x \in A$, let $\{B_n : n \in \mathbb{N}\}$ be a countable basis of neighbourhoods of x in X. Then, $\{B_n \cap A : n \in \mathbb{N}\}$ is a countable basis of neighbourhoods of x in A. Hence, A is first countable.

Let $\{X_\alpha : \alpha \in J\}$ be a family of first countable spaces and let $X = \prod_{\alpha \in J} X_\alpha$ with the product topology. For each α, let $\mathcal{B}_\alpha = \{B_{\alpha,n} : n \in \mathbb{N}\}$ be a countable basis of neighbourhoods at $x_\alpha \in X_\alpha$. Let $x = (x_\alpha) \in X$. Consider the collection

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{k} \pi_{\alpha_i}^{-1}(B_{\alpha_i,n_i}) : k \in \mathbb{N}, \alpha_1, \ldots, \alpha_k \in J, n_i \in \mathbb{N} \right\}.$$

Then, $\mathcal{B}$ is a countable basis of neighbourhoods at x in X. To see this, let U be any neighbourhood of x in X. Then, there exist finitely many indices $\alpha_1, \ldots, \alpha_k$ and neighbourhoods U_{α_i} of x_{α_i} in X_{α_i} such that

$$V = \bigcap_{i=1}^{k} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U.$$

Since $\mathcal{B}_{\alpha_i}$ is a basis at x_{α_i}, there exists n_i such that $B_{\alpha_i,n_i} \subseteq U_{\alpha_i}$. Then,

$$\bigcap_{i=1}^{k} \pi_{\alpha_i}^{-1}(B_{\alpha_i,n_i}) \subseteq V \subseteq U.$$

Hence, X is first countable. □

Definition 5.14. A subset A of a topological space X is *dense* if $\overline{A} = X$. A space X is *separable* if it contains a countable dense subset.

Example 5.15. $\mathbb{R}$ is separable: $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

$\mathbb{Q}^2$ is also a dense subset of $\mathbb{R}^2$, so $\mathbb{R}^2$ is separable.

Theorem 5.16. *For a second countable space X,*

- *(Lindelöf property) every open cover of X has a countable subcover;*
- *X is separable.*

Proof. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis of X.

- Let $\mathcal{U}$ be an open cover of X. For each B_n, choose $U_n \in \mathcal{U}$ such that $B_n \subseteq U_n$ if such U_n exists (when exists, there could be multiple such U_n, and we just choose one). Then $\{U_n : n \in \mathbb{N}, B_n \subseteq U_n\}$ is a countable collection. We claim that it covers X. For any $x \in X$, there exists $U \in \mathcal{U}$ such that $x \in U$. Since $\mathcal{B}$ is a basis, there exists some B_m such that $x \in B_m \subseteq U$. By construction, there exists $U_m \in \mathcal{U}$ such that $B_m \subseteq U_m$. Hence, $x \in U_m$. This shows that $\{U_n : n \in \mathbb{N}, B_n \subseteq U_n\}$ covers X.

- For each B_n, choose $x_n \in B_n$. Then $A = \{x_n : n \in \mathbb{N}, x_n \text{ is chosen from } B_n\}$ is a countable dense subset of X. To see this, pick any $x \in X$. Then, for any basis element B_n containing x, we have $B_n \cap A \neq \varnothing$. Hence $x \in \overline{A}$.

□

5.2 Separation Axioms

We've seen such an example before and I claimed that this appears because the points are not "separated" enough.

Example 5.17. Let $X = \{a, b\}$ with the topology $\tau = \{\emptyset, \{a\}, X\}$. Then the constant sequence (x_n) with $x_n = a$ for all $n \in \mathbb{N}$ converges to a and b.

Let's see some different ways to separate points and sets in a topological space and check if they can help us to avoid such pathological examples.

Definition 5.18. We say a topological space X is *Hausdorff* if for any two distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Lemma 5.19. *Let X be a Hausdorff space. Then, one-point sets are closed.*

Proof. Let $x \in X$. To show that $\{x\}$ is closed, we show that its complement $X \setminus \{x\}$ is open. For any $y \in X \setminus \{x\}$, since X is Hausdorff, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. Then $V \subseteq X \setminus \{x\}$. Hence for each point in $X \setminus \{x\}$, we can find an open neighbourhood contained in $X \setminus \{x\}$. This shows that $X \setminus \{x\}$ is open. □

Example 5.20. The trivial topology on a set of more than one point is not Hausdorff.

Example 5.21. Consider the space $X = \{a, b\}$ with the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$. This space is also not Hausdorff.

Example 5.22. Let $X = \mathbb{R}$. Then, complement finite and complement countable topologies on X are not Hausdorff. This is because any two non-empty open sets intersect.

Example 5.23. Every metric space is Hausdorff. For $x, y \in X$ with $x \neq y$, let $\varepsilon = d(x, y)/2 > 0$. Then the open balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint neighbourhoods of x and y, respectively.

Proposition 5.24. *Every sequence has a unique limit in a topological space.*

Proof. When X is Hausdorff, suppose that a sequence (x_n) converges to both x and y with $x \neq y$. Since X is Hausdorff, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. Since $(x_n) \to x$, there exists N_1 such that for all $n \geq N_1$, we have $x_n \in U$. Similarly, since $(x_n) \to y$, there exists N_2 such that for all $n \geq N_2$, we have $x_n \in V$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we have $x_n \in U$ and $x_n \in V$, contradicting the fact that U and V are disjoint. Hence, the limit is unique. □

Example 5.25. Let $X = \mathbb{R}$ be endowed with the *countable complement topology*. We know that any sequence (x_n) converging to x is eventually equal to x. Hence, every converging sequence has a unique limit. However, X is not Hausdorff.

Theorem 5.26. *Suppose X is first countable. Assume that every convergent sequence has a unique limit in X. Then, X is Hausdorff. (This is homework)*

In fact, metric spaces satisfy stronger separation axioms. We will introduce some of them.

Definition 5.27. Let X be a topological space such that one-point sets are closed. We say that X is:

1. *regular* if for any point x and a disjoint closed set B, there exist disjoint open sets U and V such that $x \in U$ and $B \subseteq V$.

2. *normal* if for any two disjoint closed sets A and B, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Remark 5.28. A regular space is Hausdorff. A normal space is regular (hence Hausdorff).

Example 5.29 (The K–topology on $\mathbb{R}$)**.** Let $K = \{1/n : n \in \mathbb{N}\}$ and let $\mathbb{R}_K$ be $\mathbb{R}$ with the topology generated by the basis $\{(a, b), (a, b) \setminus K : a < b\}$.

- It is Hausdorff: for $x \neq y$ pick disjoint usual intervals around x and y.

- K is closed (note K is not closed in the usual topology): if $x \notin K$ and $x \neq 0$, choose $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \cap K = \varnothing$; if $x = 0$, then $(-\varepsilon, \varepsilon) \setminus K$ is an open neighbourhood of 0 disjoint from K. Hence $\mathbb{R} \setminus K$ is open.

- It is not regular: $0 \notin K$. Suppose there are disjoint open sets U, V with $0 \in U$ and $K \subseteq V$. Choose a basis element $(a, b) \setminus K \subset U$ containing 0. Pick n large enough so that $1/n \in (a, b)$ and let (c, d) be a basis element with $1/n \in (c, d) \subseteq V$. Finally, choose z so that $z \in (\max(c, \frac{1}{n+1}), 1/n)$. Then, $z \in U \cap V$, contradicting the assumption that U and V are disjoint.

Consequently $\mathbb{R}_K$ is Hausdorff but not regular.

Now we have some other useful ways of stating these properties.

Lemma 5.30. *Let X be a topological space with one-point sets closed. Then,*

1. *X is regular iff for every neighbourhood U of a any point x, there exists a neighbourhood V of x such that $x \in V \subset \overline{V} \subset U$.*

2. *X is normal iff for every closed set A and any open set U such that $A \subset U \subseteq X$, there exists an open V with $A \subset V \subset \overline{V} \subset U$.*

Proof. We prove the regular case. The normal case is similar.

Suppose X is regular. Let U be a neighbourhood of x. Then, $X \setminus U$ is closed and disjoint from $\{x\}$. By regularity, there exist disjoint open sets V and W such that $x \in V$ and $X \setminus U \subseteq W$. Since $V \cap W = \emptyset$, we have that $V \subset X \setminus W$ which is closed. Then, $\overline{V} \subseteq X \setminus W \subseteq U$. Hence, we have found a neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$.

Conversely, let $U = X \setminus B$, where B is a closed set disjoint from $\{x\}$. By assumption, there exists a neighbourhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. So $x \in V$ and $X \setminus \overline{V} \supset X \setminus U = B$ are disjoint open sets separating x and B. Hence, X is regular. $\square$

Let me state some important results about these separation axioms.

Theorem 5.31. *Subspaces and products of Hausdorff (resp. regular) spaces are Hausdorff (resp. regular).*

Proof. Let's prove the regular case.

Let $Y \subset X$. Pick any $x \in Y$, and let B be a closed set in Y disjoint from $\{x\}$. Then, there exists a closed set C in X such that $B = Y \cap C$. Since X is regular, there exist disjoint open sets U and V in X such that $x \in U$ and $C \subseteq V$. Then, $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y such that $x \in U \cap Y$ and $B \subseteq V \cap Y$. Hence, Y is regular.

Now let $X = \prod X_\alpha$ be a product of regular spaces X_α. Let $x = (x_\alpha) \in X$ and let $U = \prod U_\alpha$ be a neighbourhood of x in X such that $U_\alpha = X_\alpha$ for all but finitely many α. By Lemma 5.30, for each α with $U_\alpha \neq X_\alpha$, there exists a neighbourhood V_α of x_α such that $x_\alpha \in V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. For other α, let $V_\alpha = X_\alpha$. Let $V = \prod V_\alpha$. Then, V is a neighbourhood of x in X such that $x \in V \subseteq \overline{V} = \prod \overline{V}_\alpha \subseteq U$. $\square$

The above theorem doesn't hold for normal spaces. Here are counterexamples.

Example 5.32. $\mathbb{R}^\mathbb{R}$ is not normal (See Munkres Chapter 4 section 32 problem 9.)

Since $\mathbb{R} \cong (0,1)$, we have that $(0,1)^\mathbb{R}$ is not normal. But $(0,1)^\mathbb{R}$ is a subspace of $[0,1]^\mathbb{R}$ which is compact Hausdorff (we will prove this later) and hence normal. So a subspace of a normal space need not be normal.

Example 5.33. Let $\mathbb{R}_\ell$ be $\mathbb{R}$ endowed with the so-called lower limit topology which is generated by the basis $\{[a,b)\}$. You can check that this is finer than the usual topology on $\mathbb{R}$ since every open interval (a,b) can be written as the union of basis elements: $(a,b) = \bigcup_{n=1}^\infty [a + \frac{1}{n}, b)$.

Then, $\mathbb{R}_\ell$ is normal: let A and B be disjoint closed sets in $\mathbb{R}_\ell$. For each $a \in A$, choose a basis element such that $[a, x_a) \cap B = \varnothing$. Similarly, for each $b \in B$, choose a basis element such that $[b, y_b) \cap A = \varnothing$. Let $U = \bigcup_{a \in A}[a, x_a)$ and $V = \bigcup_{b \in B}[b, y_b)$. Then, U and V are disjoint open sets separating A and B.

By the theorem above, we know that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is regular. But it is not normal. Indeed, $L = \{(x, -x) : x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$ and hence closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Note $\{(a, -a)\} = L \cap [a, a+1) \times [-a, -a+1)$ and hence $\{(a, -a)\}$ is open in L. So L has the discrete topology. So every subset of L is closed in subspace topology and hence closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

Now let $A = \{(x, -x) : x \in \mathbb{Q}\}$ and $B = L \setminus A$. Both are closed in $\mathbb{R}_\ell \times \mathbb{R}_\ell$. One can check that there are no disjoint open sets separating A and B (see Munkres Chapter 4 section 31 problem 9). So $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal.

There is still a positive result about subspaces of normal spaces.

Proposition 5.34. *Every closed subsets of a normal space is normal.*

Normal spaces

Theorem 5.35. *Every regular second countable space is normal.*

Proof. Let X be a regular second countable space. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis of X.

Let A and B be disjoint closed sets in X. We want to find disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

For each $a \in A$, there exists a neighbourhood U_a of a disjoint from B. By Theorem 5.30 above, there exists a neighbourhood V_a of a such that $a \in V_a \subseteq \overline{V_a} \subseteq U_a$. We then further choose a basis element B_{n_a} such that $a \in B_{n_a} \subseteq V_a$. So, we have a countable open cover $\{U_n\}$ of A so that the closure of each element is disjoint from B. Similarly, we can find a countable open cover $\{V_n\}$ of B with the property that the closure of each element is disjoint from A.

Consider the sets $U = \cup U_n$ and $V = \cup V_n$. They are not necessarily disjoint. To fix this, we modify the sets as follows:

For each n, let

$$U'_n = U_n \setminus \bigcup_{m=1}^{n} \overline{V_m}, \quad V'_n = V_n \setminus \bigcup_{m=1}^{n} \overline{U_m}.$$

Note that U'_n and V'_n are still open sets since they are obtained by removing closed sets from open sets. Also, we have $A \subseteq \bigcup_n U'_n$ and $B \subseteq \bigcup_n V'_n$ since we only removed closures of sets disjoint from A and B respectively.

Now we let

$$U' = \bigcup_n U'_n, \quad V' = \bigcup_n V'_n.$$

Then, U' and V' are open sets with $A \subseteq U'$ and $B \subseteq V'$. We claim that $U' \cap V' = \varnothing$. If not, there exists $x \in U' \cap V'$ and thence there exist U'_k and V'_j such that $x \in U'_k$ and $x \in V'_j$. Suppose WLOG that $k \geq j$. Then, by the definition of U'_k, we have

$$x \in U'_k \subseteq U_k \setminus \bigcup_{m=1}^{k} \overline{V_m} \subseteq U_k \setminus \overline{V_j}.$$

This contradicts the fact that $x \in V_j' \subseteq V_j \subseteq \overline{V_j}$. Hence, U' and V' are disjoint open sets separating A and B. □

Theorem 5.36. *Every metric space (and hence every metrizable space) is normal.*

Proof. Since metric spaces are Hausdorff, one-point sets are closed.

Let A and B be disjoint closed sets in a metric space (X, d). For each $a \in A$, there exists $r_a > 0$ such that $B_{r_a}(a) \cap B = \varnothing$ since B is closed (otherwise, $a \in \overline{B} = B$). Similarly, for each $b \in B$, there exists $r_b > 0$ such that $B_{r_b}(b) \cap A = \varnothing$.

Let $U = \bigcup_{a \in A} B_{r_a/2}(a)$ and $V = \bigcup_{b \in B} B_{r_b/2}(b)$. Then U and V are open sets with $A \subseteq U$ and $B \subseteq V$. We claim that $U \cap V = \varnothing$. If not, there exists $x \in U \cap V$. Then there exist $a \in A$ and $b \in B$ such that $x \in B_{r_a/2}(a)$ and $x \in B_{r_b/2}(b)$. WLOG, assume $r_a \leq r_b$. By the triangle inequality, we have

$$d(a, b) \leq d(a, x) + d(x, b) < \frac{r_a}{2} + \frac{r_b}{2} \leq r_b.$$

This contradicts the choice of r_b. Hence U and V are disjoint open sets separating A and B. □