Urysohn's lemma

Here is one deep theorem about normal spaces which has many important applications.

Lemma 5.37 (Urysohn's lemma)**.** *Let X be a normal space. Then for every two disjoint closed sets A and B, there exists a continuous function $f : X \to [0,1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Proof. Order the countable set of rationals in $[0,1]$ as $\{r_n : n \in \mathbb{N}\}$ with $r_1 = 0$ and $r_2 = 1$. For example, we can take the enumeration

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \dots$$

By normality, we can find disjoint open sets $U_0 \supseteq A$ and $V_0 \supseteq B$. Then, $U_0 \subseteq X \setminus V_0$ which is closed. Hence, $\overline{U_0} \subseteq X \setminus V_0 \subseteq X \setminus B$.

Let $U_1 := X \setminus B$. Then, $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$.

Continue inductively, obtaining open sets U_{r_n} for each rational $r_n \in [0,1]$ such that

$$r_n < r_m \implies \overline{U}_{r_n} \subseteq U_{r_m}$$

More precisely, suppose we have defined $U_{r_1}, \dots, U_{r_k}$. Let r_{k+1} be the next rational in the enumeration. Assume that $r_i < r_{k+1} < r_j$ for some $i, j \leq k$ and there is nothing between r_i and r_j in $\{r_1, \dots, r_k\}$. By normality, we can find disjoint open sets $\overline{U}_{r_i} \subseteq W$ and $X \setminus U_{r_j} \subseteq V$. Let $U_{r_{k+1}} = W$. Then, we have

$$\overline{U}_{r_i} \subseteq U_{r_{k+1}} \subseteq \overline{U}_{r_{k+1}} \subseteq U_{r_j}.$$

Now, for each $p \in \mathbb{Q} \cap (-\infty, 0)$, we define $U_p := \emptyset$ and for each $p \in \mathbb{Q} \cap (1, +\infty)$, we define $U_p := X$.

Finally, we define a function $f : X \to [0,1]$ by

$$f(x) := \inf\{r \in \mathbb{Q} : x \in U_r\}.$$

Note that if $x \in A$, then $x \in U_p$ for all $p \geq 0$, so $f(x) = 0$. If $x \in B$, then $x \notin U_p$ for all $p < 1$, so $f(x) = 1$.

We now check that f is continuous. Let $x_0 \in X$ and open (c,d) containing $f(x_0)$. Pick rationals p and q such that $c < p < f(x_0) < q < d$ (This is why we need the weird extension before).

We claim that $U_q - \overline{U}_p$ is an open neighbourhood of x_0 with $f(U_q - \overline{U}_p) \subseteq (c,d)$.

Note that $f(x_0) < q$ implies $x_0 \in U_q$. Also, $f(x_0) > p$ implies $x_0 \notin \overline{U}_p$. Hence, $x_0 \in U_q - \overline{U}_p$.

Finally, let $x \in U_q - \overline{U}_p$. Then, $x \in U_q$ and $x \notin \overline{U}_p$. Thus, $f(x) \leq q$ and $f(x) \geq p$, which shows that $f(x) \in [p,q] \subset (c,d)$. ☐

5.3 Urysohn's Metrization Theorem

Theorem 5.38 (Urysohn's metrization theorem)**.** *Every regular space with a countable base is metrizable.*

We first prove the following lemma.

Lemma 5.39. *Let X be a regular space with a countable basis. Then, there exists a countable family of continuous functions $f_n : X \to [0,1]$ such that for any $x \in X$ and any neighborhood U of x, there exists n with $f_n(x) > 0$ and $f_n|_{U^c} \equiv 0$.*

Proof. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable basis of X. Let $\mathcal{S} := \{(B_n, B_m) \in \mathcal{B} \times \mathcal{B} : \overline{B}_n \subseteq B_m\}$. For each $(B_n, B_m) \in \mathcal{S}$, by Urysohn's lemma, there exists a continuous function $f_{n,m} : X \to [0,1]$ such that $f_{n,m}|_{\overline{B}_n} \equiv 1$ and $f_{n,m}|_{B_m^c} \equiv 0$.

Then, for any $x \in X$ and any neighborhood U of x, there exists $B_m \in \mathcal{B}$ such that $x \in B_m \subseteq U$. By regularity, there exists $B_n \in \mathcal{B}$ such that $x \in \overline{B}_n \subseteq B_m$. Then, $f_{n,m}$ satisfies the requirement. □

Proof of Theorem 5.38. We reindex the functions in the lemma above as $\{f_n : n \in \mathbb{N}\}$. Define a map $F : X \to [0,1]^{\mathbb{N}}$ by $F(x) = (f_1(x), f_2(x), \ldots)$.

Since each f_n is continuous, F is continuous when $[0,1]^{\mathbb{N}}$ is equipped with the product topology (cf. Theorem 3.91).

We claim that F is an embedding. To see this, we first show that F is injective. If $x, y \in X$ with $x \neq y$, then there exists a neighborhood U of x such that $y \notin U$. By the lemma, there exists n such that $f_n(x) > 0$ and $f_n|_{U^c} \equiv 0$. Hence, $f_n(y) = 0 \neq f_n(x)$, which shows that $F(x) \neq F(y)$.

Next, we show that F is a homeomorphism onto its image. Since we already showed that F is continuous and injective, it suffices to show that $F(U)$ is open in $F(X)$ for any open set U.

Let x be any point in U and hence $f(x)$ is an arbitrary point in $F(U)$. By the construction of $\{f_n\}$, there exists n such that $f_n(x) > 0$ and $f_n|_{U^c} \equiv 0$. Let $\pi_n : [0,1]^{\mathbb{N}} \to [0,1]$ be the projection onto the n-th coordinate. Then, $\pi_n^{-1}((f_n(x)/2, 1]) \cap F(X)$ is an open neighborhood of $F(x)$ in $F(X)$ and

$$\pi_n^{-1}((f_n(x)/2, 1]) \cap F(X) = F(\{y \in X : f_n(y) > f_n(x)/2 > 0\}) \subseteq F(U).$$

This shows that $F(U)$ is open in $F(X)$.

Now, we recall that $[0,1]^{\mathbb{N}}$ with the product topology is metrizable (Homework). Hence, $F(X)$ is metrizable as a subspace of a metric space. Since F is an embedding, X is metrizable. □

6 Compactness

Let's have a look at two examples from calculus:

Theorem 6.1 (Intermediate Value Theorem)**.** *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. WLOG, we assume that $f(0) < f(1)$. Then, for any $r \in [f(0), f(1)]$, there exists $c \in (0,1)$ such that $f(c) = r$.*

Theorem 6.2 (Maximum value theorem)**.** *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [0,1]$ such that $f(c) \geq f(x)$ for all $x \in [0,1]$.*

These are due to the fact that $[0,1]$ is "connected" and "compact". We will talk about these two notions for general topological spaces in this and the next section.

Definition 6.3. A topological space X is *compact* if every open cover $\mathcal{U}$ of X has a finite subcover, i.e., there exist finitely many $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $X \subseteq U_1 \cup U_2 \cup \ldots \cup U_n$.

Example 6.4. $\mathbb{R}$ is not compact: $\{(n-1, n+1)\}_{n\in\mathbb{Z}}$ is an open cover of $\mathbb{R}$ with no finite subcover.

Example 6.5. Every finite topological space X is compact (regardless of the topology): given any open cover $\mathcal{U}$, for each point $x \in X$ choose $U_x \in \mathcal{U}$ such that $x \in U_x$. Then, $\{U_x : x \in X\}$ is a finite subcover.

Example 6.6. $(0,1]$ is not compact: $\{(1/n, 1] : n \in \mathbb{N}\}$ is an open cover of $(0,1]$ with no finite subcover.

Example 6.7. Any closed interval such as $[0,1]$ in real line is compact. This is not trivial and we will prove it later. As a consequence, we will see $X \subset \mathbb{R}^n$ is compact iff it is closed and bounded (Heine-Borel theorem).

Although we have the result for $\mathbb{R}^n$, not every closed and bounded subset of a general metric space is compact. Let me show you two examples.

Example 6.8. Let $X = [0,1]$ with the discrete topology. Then, X is closed and bounded as a subset of itself. However, it is not compact: $\{\{x\} : x \in [0,1]\}$ is an open cover of X with no finite subcover.

We say a collection of sets $\{U_\alpha\}$ in X covers $Y \subset X$ if $Y \subseteq \bigcup_\alpha U_\alpha$.

Lemma 6.9. *$Y \subset X$ is compact iff every cover of Y by open sets in X has a finite subcover.*

Proof. Y is compact iff every open cover of the form $\{U_\alpha \cap Y\}$ has a finite subcover iff every cover $\{U_\alpha\}$ of Y by open sets in X has a finite subcover. □

Example 6.10. $\{1/n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is not compact: $\{\{1/n\}\}$ is an open cover with no finite subcover.

However, $\{0\} \cup \{1/n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is compact: given any open cover $\mathcal{U}$, there exists $U \in \mathcal{U}$ such that $0 \in U$. Choose $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset U$. Then, only finitely many points of the set $\{1/n\}_{n\in\mathbb{N}}$ are outside U. Choosing open sets from $\mathcal{U}$ covering these finitely many points gives a finite subcover.

Theorem 6.11. *Every closed subset Y of a compact space X is compact.*

Proof. Let $\{U_\alpha\}$ be an open cover of Y by open sets in X. Then, $\{U_\alpha\} \cup \{X \setminus Y\}$ is an open cover of X. By compactness of X, there exists a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}, X \setminus Y\}$ of X. Then, $Y \subseteq \bigcup_{i=1}^n U_{\alpha_i}$, we conclude that $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of Y. $\square$

Theorem 6.12. *Every compact subspace Y of a Hausdorff space X is closed.*

Proof. We will show that $X \setminus Y$ is open. Given any fixed $x \in X \setminus Y$, for each $y \in Y$ choose disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Then, $\{V_y : y \in Y\}$ is an open cover of Y. By compactness of Y, there exists a finite subcover $\{V_{y_1}, \dots, V_{y_n}\}$ of Y. Let $U = \bigcap_{i=1}^n U_{y_i}$. Then, U is disjoint from $V_{y_1} \cup \dots \cup V_{y_n}$ and hence is an open neighbourhood of x disjoint from Y. Hence, $X \setminus Y$ is open. $\square$

In fact, the proof above shows that for any $x \in X \setminus Y$, there exist disjoint open sets U and V such that $x \in U$ and $Y \subseteq V$.

Example 6.13. This theorem immediately tells us $(a, b]$ or $[a, b)$ is not compact in $\mathbb{R}$ as it is not closed.

Note that the compactness assumption is necessary.

Example 6.14. Let $X = \mathbb{R}$ be endowed with the finite complement topology. Then, X is not Hausdorff as we have seen. However, any subset Y of X is compact.

Theorem 6.15. *Every compact Hausdorff space is normal.*

Proof. Let A and B be disjoint closed sets in a compact Hausdorff space X. Then, they are compact as well. For each $a \in A$, choose disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Then, $\{U_a : a \in A\}$ is an open cover of A. By compactness of A, there exists a finite subcover $\{U_{a_1}, \dots, U_{a_n}\}$ of A. Let $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$. Then, U and V are disjoint open sets separating A and B. $\square$

Finite intersection property

Definition 6.16. A collection of sets $\{C_\alpha\}$ in X is said to have the *finite intersection property* if for any finite subcollection $\{C_{\alpha_1}, \dots, C_{\alpha_n}\}$, we have $\bigcap_{i=1}^n C_{\alpha_i} \neq \emptyset$.

Theorem 6.17. *A topological space X is compact iff for any collection of closed sets $\{C_\alpha\}$ in X with the finite intersection property, we have $\bigcap_\alpha C_\alpha \neq \emptyset$.*

Proof. ($\Rightarrow$) Let $\{C_\alpha\}$ be a collection of closed sets in X with the finite intersection property. Suppose that $\bigcap_\alpha C_\alpha = \emptyset$. Then, $\{X \setminus C_\alpha\}$ is an open cover of X. By compactness of X, there exists a finite subcover $\{X \setminus C_{\alpha_1}, \dots, X \setminus C_{\alpha_n}\}$ of X. This implies that $\bigcap_{i=1}^n C_{\alpha_i} = \emptyset$, contradicting the finite intersection property.

($\Leftarrow$) Let $\{U_\alpha\}$ be an open cover of X. Suppose that there is no finite subcover. Then, for any finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, we have $\bigcap X \setminus U_{\alpha_i} = X \setminus (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}) \neq \emptyset$. This implies that the collection of closed sets $\{X \setminus U_\alpha\}$ has the finite intersection property. By assumption, we have $\bigcap_\alpha (X \setminus U_\alpha) \neq \emptyset$, contradicting the fact that $\{U_\alpha\}$ is an open cover of X. $\square$

Relation with continuous functions

Theorem 6.18. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$ by open sets in Y. Then, $\{f^{-1}(V_\alpha)\}$ is an open cover of X by open sets in X. By compactness of X, there exists a finite subcover $\{f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_n})\}$ of X. Then, $f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$, we conclude that $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ is a finite subcover of $f(X)$. $\square$

As a consequence we can go back to the maximum value theorem.

Corollary 6.19 (Maximum value theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous function. Then there exists $c \in X$ such that $f(c) \geq f(x)$ for all $x \in X$.*

Proof. By the previous theorem, $f(X)$ is compact in $\mathbb{R}$. In particular, it is closed and bounded. Let $M = \sup f(X)$. This implies that there will be a sequence $\{y_n\} \subset \mathbb{R}$ such that $y_n \to M$. Since $f(X)$ is closed, the limit $M \in f(X)$. Then, it is not hard to see that $M \in f(X)$. Hence, there exists $c \in X$ such that $f(c) = M \geq f(x)$ for all $x \in X$. $\square$

Theorem 6.20. *Let $f : X \to Y$ be a bijective continuous map. Assume that X is compact and Y is Hausdorff. Then f is a homeomorphism.*

Proof. It suffices to show that f is a closed map. Given any closed set C in X, it is compact. By continuity of f, $f(C)$ is compact as well. By Hausdorffness of Y, $f(C)$ is closed in Y. Hence, f is a homeomorphism. $\square$

Finite vs Compact

Compactness can be viewed as a generalization of finiteness. In particular, compactness to continuous maps is what finiteness is to functions.

Proposition 6.21. *Any function sends a finite set to a finite set.*
Any continuous map sends a compact set to a compact set.

Theorem 6.22.
- *Any finite function $f : A \to \mathbb{R}$ obtains maximal and minimal values.*
- *Any continuous function $f : X \to \mathbb{R}$ from a compact space X obtains maximal and minimal values.*

Theorem 6.23.
- *Any sequence (x_n) taking value in a finite set A has a convergent subsequence (actually constant).*
- *Any sequence (x_n) taking value in a compact* metric space *X has a convergent subsequence.*

6.1 Order topology and Metric spaces

Theorem 6.24. *Let X be a linear post. Assume that X has the least upper bound property*[1]*. Then, any closed interval $[a, b]$ in X is compact under the order topology (generated by (a, b) and $[a_0, b)$ and $(a, b_0]$ if a_0 is the least element and b_0 is the greatest element).*

Proof. Let $\mathcal{U}$ be an open cover of $[a, b]$ where $a < b$. Define

$$S = \{x \in [a, b] : [a, x] \text{ can be covered by a finite subcover of } \mathcal{U}\}.$$

It is clear that $a \in S$ and if $x \in S$, then $[a, x] \subset S$. Since S is clearly upper-bounded by b, we let $c = \sup S$, i.e., the least upper bound of S. We will show that $c \in S$ and $c = b$.

Clearly $c \leq b$. If (a, c) is empty, then $a = c \in S$. Suppose (a, c) is nonempty. Then, we have that $[a, c) \subset S$. Let U_c be an open set in $\mathcal{U}$ containing c. By the definition of the order topology, there exists some (d, e) such that $c \in (d, e) \subseteq U_c$. There must be some $x \in (d, c)$ otherwise $\sup S \leq d < c$. Then, we know $x \in S$ and hence $[a, x]$ can be covered by a finite subcover $\mathcal{U}_x$ of $\mathcal{U}$. Note that $\mathcal{U}_x \cup \{U_c\}$ is a finite subcover of $[a, c]$. This implies that $c \in S$.

Suppose $c < b$. Now we consider the interval (c, b). Let $T = \{x \in [c, b] : [c, x] = \{c\}\}$ and let $y = \sup T$. Note that $y \leq b$. If $y = c$, then the open set U_c containing c also contains some element z of (c, b), and hence $[x, z]$ can be covered by a finite subcover, which contradicts the definition of c. Therefore, we must have $y > c$. But then the interval $[a, y]$ can be covered by a finite subcover of $\mathcal{U}$ by considering an open set around y, contradicting the definition of c. Hence, we must have $c = b$.

Therefore, we have shown that every open cover $\mathcal{U}$ of $[a, b]$ has a finite subcover. This completes the proof. □

As a corollary, we have the following result.

Corollary 6.25. *Any closed and bounded interval $[a, b]$ in $\mathbb{R}$ is compact.*

We can now use this corollary to prove the Heine-Borel theorem.

Theorem 6.26 (Heine-Borel theorem)**.** *A subset K of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. ($\Rightarrow$) Assume K is compact. Then, K is closed since $\mathbb{R}^n$ is Hausdorff. Then, consider the collection of open sets $\{B_m(0) : m \in \mathbb{N}\}$, which is a cover of $\mathbb{R}^n$ and hence K. There exists a finite subcover $\{B_{m_1}(0), B_{m_2}(0), \ldots, B_{m_k}(0)\}$ of K. Then, let $M = \max\{m_1, m_2, \ldots, m_k\}$. It follows that $K \subseteq B_M(0)$, showing that K is bounded.

($\Leftarrow$) Assume K is closed and bounded. Since K is bounded, it is contained in some ball $B_R(0)$ for some radius $R > 0$. But then $B_R(0) \subset [-R, R]^n$. $[-R, R]^n$

[1] This means any subset that is bounded above has a least upper bound. For example, we know $\mathbb{R}$ satisfies this property as we can always take the supremum of a bounded set.

is compact as it is a finite product of compact sets (which we will prove later). So K as a closed subset of a compact set is compact. $\square$

In general metric spaces, we can't expect closed and bounded subsets to be compact as we have seen before. However, we do have the following important characterizations of compactness in metric spaces.

Definition 6.27. Let (X, d) be a metric space. A sequence (x_n) is said to be *Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

The metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Example 6.28. The Euclidean space $\mathbb{R}^n$ is a complete metric space. $\mathbb{R}^n \setminus \{0\}$ is not complete.

Definition 6.29. A metric space (X, d) is said to be *totally bounded* if for every $\epsilon > 0$, there exists a finite set $F \subset X$ such that X is covered by the open balls $B_\epsilon(x)$ for $x \in F$.

Theorem 6.30. *A metric space is compact iff it is complete and totally bounded.*

We will postpone the proof of this theorem until we have developed more tools.

We now prove one thing that we used previously in showing $\S^1$ is not contractible.

Definition 6.31. A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be *uniformly continuous* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.

Theorem 6.32. *Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous function between two metric spaces. If X is compact, then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, for each $x \in X$, by continuity of f at x, there exists $\delta_x > 0$ such that for all $y \in X$, if $d_X(x, y) < \delta_x$, then $d_Y(f(x), f(y)) < \epsilon/2$. Then, $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X. By compactness of X, there exists a finite subcover $\{B_{\delta_{x_i}/2}(x_i) : i = 1, 2, \ldots, n\}$. Let $\delta = \min\{\delta_{x_i}/2 : i = 1, 2, \ldots, n\}$. We will show that this δ works. Given any $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, there exists some i such that $x_1 \in B_{\delta_{x_i}/2}(x_i)$. This implies that $d_X(x_1, x_i) < \delta_{x_i}/2$ and hence $d_X(x_2, x_i) \leq d_X(x_2, x_1) + d_X(x_1, x_i) < \delta_{x_i}/2 + \delta_{x_i}/2 = \delta_{x_i}$. Therefore, we have

$$d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f(x_i)) + d_Y(f(x_i), f(x_2)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

$\square$