6.1.1 Compactness via sequences and nets

Definition 6.33. A space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

Theorem 6.34. *Let X be a metric space. Then, the following are equivalent:*

1. *X is compact.*
2. *X is complete and totally bounded.*
3. *X is sequentially compact.*

Proof. (3) $\Rightarrow$ (2): For completeness, let (x_n) be a Cauchy sequence in X. By sequential compactness, there exists a convergent subsequence (x_{n_k}) converging to some $x \in X$. Since (x_n) is Cauchy, it must converge to the same limit x (exercise). Hence, X is complete. For total boundedness, suppose not. Then, there exists $\epsilon > 0$ such that for any finite set $F \subset X$, there exists some $x \in X$ such that $d(x, y) \geq \epsilon$ for all $y \in F$. In this way, we can construct a sequence (x_n) in X such that $d(x_n, x_m) \geq \epsilon$ for all $n \neq m$. This sequence has no convergent subsequence, contradicting the sequential compactness of X. Hence, X is totally bounded.

(2) $\Rightarrow$ (1): Assume X is complete and totally bounded. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Suppose there is no finite subcover.

Using total boundedness, for each $n \geq 1$ choose a finite 2^{-n}-net $F_n \subset X$. We build inductively a sequence (x_n) with (i) $x_n \in F_n$, (ii) no finite subfamily of $\{U_\alpha\}$ covers $B_{2^{-n}}(x_n)$, and (iii) for $n \geq 2$, $B_{2^{-(n-1)}}(x_{n-1}) \cap B_{2^{-n}}(x_n) \neq \emptyset$.

Base step: pick $x_1 \in F_1$ with (ii); such a choice exists because if every ball $\{B_{2^{-1}}(y) : y \in F_1\}$ had a finite subcover, then, since F_1 is finite, X would have a finite subcover. Induction: having chosen x_{n-1}, consider those $y \in F_n$ with $B_{2^{-(n-1)}}(x_{n-1}) \cap B_{2^{-n}}(y) \neq \emptyset$. If each such ball had a finite subcover, finitely many of them would cover $B_{2^{-(n-1)}}(x_{n-1})$, contradicting (ii) for x_{n-1}. Hence choose x_n with (ii)–(iii).

From (iii), pick $z \in B_{2^{-(n-1)}}(x_{n-1}) \cap B_{2^{-n}}(x_n)$; then $d(x_{n-1}, x_n) \leq d(x_{n-1}, z) + d(z, x_n) \leq 2^{-(n-1)} + 2^{-n} \leq 2^{-(n-2)}$. Thus for $m < n$,

$$d(x_m, x_n) \leq \sum_{k=m+1}^{n} d(x_{k-1}, x_k) \leq \sum_{k=m+1}^{n} 2^{-(k-2)} \leq 2^{-(m-2)},$$

so (x_n) is Cauchy and, by completeness, $x_n \to a \in X$.

Choose α_0 with $a \in U_{\alpha_0}$ and $\varepsilon > 0$ such that $B(a, \varepsilon) \subset U_{\alpha_0}$. Pick n with $d(x_n, a) < \varepsilon/2$ and $2^{-n} < \varepsilon/2$. Then for any $y \in B(x_n, 2^{-n})$, $d(y, a) \leq d(y, x_n) + d(x_n, a) < \varepsilon$; hence $B(x_n, 2^{-n}) \subset B(a, \varepsilon) \subset U_{\alpha_0}$, contradicting (ii). Therefore the cover has a finite subcover, and X is compact.

(1) $\Rightarrow$ (3): Assume X is compact. Let (x_n) be any sequence in X. If $\{x_n : n \in \mathbb{N}\}$ is finite, then there exists some $x \in X$ such that $x_n = x$ for infinitely many n. Hence, the constant subsequence converges to x. Now assume that $\{x_n : n \in \mathbb{N}\}$ is infinite. Let $A = \{x_n : n \in \mathbb{N}\}$. Suppose that (x_n) has

no convergent subsequence. Then, for any $x \in X \setminus A$, there exists an open neighbourhood U_x of x such that $U_x \cap A = \emptyset$. This implies that A is closed and hence compact. For any $a \in A$, since no subsequence of (x_n) converges to a, there exists an open neighbourhood V_a of a such that $V_a \cap A = \{a\}$. Then, $\{V_a\}$ is an infinite open cover of A with no finite subcover, contradicting the compactness of A. Therefore, (x_n) has a convergent subsequence.

□

Neither does sequential compactness imply compactness nor vice versa in general topological spaces.

Example 6.35 (Compact but not sequentially compact)**.** Let $X = \{0,1\}^{[0,1]}$ with the product topology. By Tychonoff, X is compact. For each $n \in \mathbb{N}$ define $f_n \in X$ by $f_n(r) =$ the n-th binary digit of r (choose the expansion not ending in all 1's). Take any subsequence (f_{n_k}). If it converges, we know that for any $r \in [0,1]$, the sequence $(f_{n_k}(r))$ must converge in $[0,1]$. Choose $r \in [0,1]$ whose binary expansion has digit n_k equal to 0 if k is odd and 1 if k is even (this is possible since we prescribe digits only on the positions $\{n_k\}$). Then $f_{n_k}(r) = 0,1,0,1,\ldots$ does not converge. Therefore (f_{n_k}) cannot converge in the product topology. Hence no subsequence of (f_n) converges; X is not sequentially compact.

Example 6.36 (Sequentially compact but not compact)**.** Let $Y \subset \{0,1\}^{[0,1]}$ be a subspace consisting of all functions f such that $f(r) = 1$ for at most countably many $r \in [0,1]$. We claim that Y is sequentially compact but not compact. Given any sequence (f_n) in Y, let $A = \bigcup_n \{r : f_n(r) = 1\}$. Then, A is countable. For each $r \in A$, the sequence $(f_n(r))$ in $\{0,1\}$ has a convergent subsequence. By a diagonal argument, we can find a subsequence (f_{n_k}) such that for each $r \in A$, the sequence $(f_{n_k}(r))$ converges in $\{0,1\}$. For $r \notin A$, define $f(r) = 0$. Then, it is easy to see that $f \in Y$ and $f_{n_k} \to f$ in the product topology. Hence, Y is sequentially compact.

To see that Y is not compact, we just need to show that Y is not closed. Pick $\mathbf{1} \in X \setminus Y$ being the constant function with value 1. For any basic open neighbourhood of $\mathbf{1}$ of the form

$$U = \bigcap_{i=1}^{n} \pi_{r_i}^{-1}(\{1\})$$

where $\pi_{r_i} : X \to \{0,1\}$ is the projection map, we can find $f \in Y$ such that $f(r_i) = 1$ for all $i = 1,2,\ldots,n$ (for example, let $f(r) = 1$ if $r = r_i$ for some i and $f(r) = 0$ otherwise). This shows that every open neighbourhood of $\mathbf{1}$ intersects Y. Hence, $\mathbf{1} \in \overline{Y} \setminus Y$ and Y is not closed, therefore not compact.

In general topological spaces, compactness can be characterized using nets. A net is a generalization of a sequence that allows for indexing by a directed set.

Definition 6.37. We say a net $(x_\alpha)_{\alpha\in A}$ has a cluster point $x \in X$ if for every open neighbourhood U of x and every $\alpha_0 \in A$, there exists (not for all) $\alpha \geq \alpha_0$ such that $x_\alpha \in U$.

One equivalent way to describe cluster points is via subnets.

Definition 6.38. A *subnet* of a net $P : \Lambda \to X$ is the composition $P \circ \varphi$ where $\varphi : \Gamma \to \Lambda$ is an increasing function between directed sets Γ and Λ such that: For every $\lambda \in \Lambda$, there exists $\gamma \in \Gamma$ such that $\lambda \leq \varphi(\gamma)$. If we represent P as (x_λ), then the subnet can be represented as (x_{λ_γ}) or $(x_{\varphi(\gamma)})$ (cf. the subsequence notation (x_{n_k})).

Example 6.39. Any subsequence (x_{n_k}) of a sequence (x_n) is a subnet. Note that here the map $\varphi : \mathbb{N} \to \mathbb{N}$ is defined by $\varphi(k) = n_k$.

Proposition 6.40. *A net $(x_\alpha)_{\alpha\in A}$ has a cluster point $x \in X$ iff there exists a subnet of $(x_\alpha)_{\alpha\in A}$ that converges to x.*

Proof. ($\Rightarrow$) Assume x is a cluster point of the net $(x_\alpha)_{\alpha\in A}$. Define the directed set

$$\Gamma = \{(\alpha, U) : \alpha \in A, U \text{ is an open neighbourhood of } x_\alpha\}$$

with the order $(\alpha_1, U_1) \leq (\alpha_2, U_2)$ if and only if $\alpha_1 \leq \alpha_2$ and $U_2 \subseteq U_1$. Define the map $\varphi : \Gamma \to A$ by $(\alpha, U) \mapsto \alpha$. Then, the subnet $(x_{\varphi(\gamma)})_{\gamma\in\Gamma}$ converges to x. (Note that the map φ is not injective, so a subnet is quite different from a subsequence.)

($\Leftarrow$) Assume there exists a subnet $(x_{\varphi(\gamma)})_{\gamma\in\Gamma}$ that converges to x. Given any open neighbourhood U of x and any $\alpha_0 \in A$, by the definition of subnet, there exists some $\gamma_0 \in \Gamma$ such that for all $\gamma \geq \gamma_0$, we have $\varphi(\gamma) \geq \alpha_0$. Since the subnet converges to x, there exists some $\gamma_1 \geq \gamma_0$ such that for all $\gamma \geq \gamma_1$, we have $x_{\varphi(\gamma)} \in U$. In particular, for this choice of γ_1, we have some $\beta = \varphi(\gamma_1) \geq \alpha_0$ such that $x_\beta = x_{\varphi(\gamma_1)} \in U$. Therefore, x is a cluster point of the net. $\square$

Theorem 6.41. *A topological space X is compact iff every net in X has a cluster point and hence iff every net in X has a convergent subnet.*

Proof. ($\Rightarrow$) Assume X is compact. Let $(x_\alpha)_{\alpha\in A}$ be any net in X. For each $\alpha \in A$, let

$$C_\alpha = \overline{\{x_\beta : \beta \geq \alpha\}}.$$

Then, each C_α is closed and the collection $\{C_\alpha\}_{\alpha\in A}$ has the finite intersection property. By compactness of X, we have

$$\bigcap_{\alpha\in A} C_\alpha \neq \emptyset.$$

Let x be any point in this intersection. We will show that x is a cluster point of the net. Given any open neighbourhood U of x and any $\alpha_0 \in A$, since $x \in C_{\alpha_0}$, we have

$$U \cap \{x_\beta : \beta \geq \alpha_0\} \neq \emptyset.$$

Hence, there exists some $\beta \geq \alpha_0$ such that $x_\beta \in U$. Therefore, x is a cluster point of the net.

($\Leftarrow$) Assume every net in X has a cluster point. Let $\{C_i\}_{i \in I}$ be a collection of closed sets in X with the finite intersection property. Direct the set $\mathcal{F}$ of finite subsets of I by inclusion, and for each $F \in \mathcal{F}$, choose

$$x_F \in \bigcap_{i \in F} C_i$$

(nonempty by the finite intersection property). The net $(x_F)_{F \in \mathcal{F}}$ has a cluster point x. Now pick any neighbourhood U of x. For any $i \in I$, consider $F_0 = \{i\}$. By definition of cluster point, there exists some $F \supseteq F_0$ such that $x_F \in U$. Since $x_F \in C_i$, we have $U \cap C_i \neq \emptyset$. This shows that every neighbourhood of x intersects C_i, so $x \in \overline{C_i} = C_i$. Hence,

$$x \in \bigcap_{i \in I} C_i.$$

Therefore, the intersection is nonempty, and X is compact. $\square$

6.2 Tychonoff's theorem

There are multiple proofs of Tychonoff's theorem. You can check Munkres book for such proofs. We will give a proof using nets here which is somewhat conceptually clean.

We know that if a net has a cluster point, then it has a subnet that converges to that cluster point. Then, is there any special kind of net such that if it has a cluster point, then it converges to that cluster point? The answer is yes, and one kind of such nets is called ultranets.

Definition 6.42. A net $\{x_\lambda\}$ in a set X is an *ultranet* if for any subset $S \subset X$, the net is eventually in S or eventually in $X \setminus S$ (there exists λ_0 such that either $x_\lambda \in S$ for all $\lambda \geq \lambda_0$ or $x_\lambda \in X \setminus S$ for all $\lambda \geq \lambda_0$).

Example 6.43. Any constant net is an ultranet. Any eventually constant net is an ultranet. If Λ has a greatest element λ_0, then any net indexed by Λ is an ultranet as it is eventually constant at x_{λ_0}.

One immediate consequence of this definition is the following lemma.

Lemma 6.44. *An ultranet converges to each of its cluster points.*

Proof. Let $\{x_\lambda\}$ be an ultranet in X and let $x \in X$ be a cluster point of the net. Given any open neighbourhood U of x, since $\{x_\lambda\}$ is an ultranet, it is eventually in U or eventually in $X \setminus U$. Since x is a cluster point, it cannot be eventually in $X \setminus U$. Hence, it is eventually in U. Therefore, $\{x_\lambda\}$ converges to x. $\square$

Lemma 6.45. *Every net has a subnet that is an ultranet.*

This is a nontrivial result and we will not give a full proof here. I'll provide a reference on Canvas if you are interested in the full proof.

Sketch of proof. Let (x_λ) be a net in a set X. We call $Q \subset P(X)$ a admissible family if

- for any $A_1, A_2 \in Q$, we have $\emptyset \neq A_1 \cap A_2 \in Q$;
- for any $A \in Q$, the net (x_λ) is frequently in A (for any λ_0, there exists some $\lambda \geq \lambda_0$ such that $x_\lambda \in A$).

Let $\mathcal{Q}$ denote the collection of all admissible families and we partially order $\mathcal{Q}$ by inclusion. Using Zorn's lemma, we can show that $\mathcal{Q}$ has a maximal element Q_{max}. You can show that if $A \subset X$, then either $A \in Q_{max}$ or $X \setminus A \in Q_{max}$. Now, we define a directed set $\Gamma = \{(\lambda, A) : \lambda \in \Lambda, A \in Q_{max},\ \text{s.t. } x_\lambda \in A\}$ with the order defined by $(\lambda_1, A_1) \leq (\lambda_2, A_2)$ if $\lambda_1 \leq \lambda_2$ and $A_2 \subseteq A_1$. Then, let $\varphi : \Gamma \to \Lambda$ be defined by $(\lambda, A) \mapsto \lambda$. This defines a subnet and you can check that this subnet is indeed an ultranet. $\square$

Lemma 6.46. *If $\{x_\lambda\}$ is an ultranet in X and $f : X \to Y$ is a map, then $\{f(x_\lambda)\}$ is an ultranet in Y.*

Proof. Given any subset $T \subset Y$, let $S = f^{-1}(T)$. Since $\{x_\lambda\}$ is an ultranet in X, it is eventually in S or eventually in $X \setminus S = f^{-1}(Y \setminus T)$. Hence, $\{f(x_\lambda)\}$ is eventually in T or eventually in $Y \setminus T$. $\square$

Theorem 6.47 (Tychonoff's theorem)**.** *A nonempty product space is compact iff each factor space is compact.*

Proof. ($\Rightarrow$) Since each projection map $\pi_\alpha : X = \prod X_\alpha \to X_\alpha$ is continuous, we have that $X_\alpha = \pi_\alpha(X)$ is compact for each α.

($\Leftarrow$) Assume that X_α is compact for each α. Let $\{x_\lambda\}$ be a net in $X = \prod X_\alpha$. Then, $\{x_\lambda\}$ has a subnet $\{x_{\lambda_\mu}\}$ that is an ultranet. Then, for each α, the net $\{\pi_\alpha(x_{\lambda_\mu})\}$ is an ultranet in X_α which has some cluster point $y_\alpha \in X_\alpha$ as X_α is compact. Since $\{\pi_\alpha(x_{\lambda_\mu})\}$ is an ultranet, it converges to y_α. Then, the net $\{x_{\lambda_\mu}\}$ converges to $y = (y_\alpha) \in X$ which is hence a cluster point of the original net $\{x_\lambda\}$. Therefore, by the characterization of compactness using nets, X is compact. $\square$

Example 6.48. $\{0,1\}^{[0,1]}$ is compact. $[0,1]^{\mathbb{R}}$ is compact.

Example 6.49. The Cantor set C is compact as C is bounded and C is closed since $C = \bigcap C_n$ where each C_n is closed.

We can also see this using Tychonoff's theorem. We know there is a bijection between C and $\{0,2\}^{\mathbb{N}}$: each $x \in C$ can be written in base 3 using only digits 0 and 2, and this gives a bijection between C and $\{0,2\}^{\mathbb{N}}$. It turns out that this bijection is actually a homeomorphism. To see this, let's actually give a metric on $\{0,2\}^{\mathbb{N}}$ that induces the product topology. For any two sequences $(a_n),(b_n) \in \{0,2\}^{\mathbb{N}}$, define

$$d((a_n),(b_n)) = \begin{cases} 0 & \text{if } a_n = b_n \text{ for all } n \\ 1/3^k & \text{where } k = \min\{n : a_n \neq b_n\} \end{cases}$$

We defined an ultrametric on C before. You can check that this metric on $\{0,2\}^{\mathbb{N}}$ is actually the same as the ultrametric on C under the above bijection. So the bijection is an isometry. The only thing left to check is that this metric on $\{0,2\}^{\mathbb{N}}$ induces the product topology. This is left as an exercise. Therefore, C is homeomorphic to $\{0,2\}^{\mathbb{N}}$ which is compact by Tychonoff's theorem. Hence, C is compact.

6.3 Local compactness and compactification

Definition 6.50. A topological space X is said to be *locally compact* if for every point $x \in X$, there exists an open neighbourhood U of x and a compact set K such that $U \subseteq K$.

Example 6.51. Any compact space is locally compact.

Example 6.52. $\mathbb{R}$ is locally compact. For each $x \in \mathbb{R}$, we pick the open neighbourhood $U = (x-1, x+1)$ and the compact set $K = [x-1, x+1]$ such that $U \subset K$.

Similarly, $\mathbb{R}^n$ is locally compact. For each $x \in \mathbb{R}^n$, we pick the open neighbourhood $U = (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1)$ and the compact set $K = [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1]$ such that $U \subset K$.

Example 6.53. $\mathbb{R}^{\mathbb{N}}$ is not locally compact. For any $x \in \mathbb{R}^{\mathbb{N}}$, pick any basic open neighborhood $U = U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$. Suppose there exists some compact set K such that $U \subset K$. Consider the projection map $\pi_{n+1} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ defined by $\pi_{n+1}((x_i)) = x_{n+1}$. Then, $\pi_{n+1}(U) = \mathbb{R}$ and hence $\pi_{n+1}(K) \supset \pi_{n+1}(U) = \mathbb{R}$. However, the continuous image of a compact set is compact and $\mathbb{R}$ is not compact. This is a contradiction. Therefore, there does not exist any compact set K such that $U \subset K$ and hence $\mathbb{R}^{\mathbb{N}}$ is not locally compact.

Theorem 6.54. *Let X be a topological space. Then, X is locally compact Hausdorff iff there exists a compact Hausdorff space Y such that*

- *X is a subspace of Y;*
- *$Y \setminus X$ consists of a single point.*

Example 6.55. $X = \mathbb{R} \cong (0,1)$ and $Y = S^1 \cong [0,1]/_{0\sim 1}$

Proof. $(\Rightarrow)$ Let $Y = X \sqcup \{\infty\}$ with the topology τ defined as follows:

- For any open set U in X, $U \in \tau$;