Definition 2.25. Dendrogram (finite case). Let X be finite. A dendrogram on X is an order preserving map $\theta : [0, \infty) \to Part(X)$ on X such that:

1. $\theta(0)$ is the discrete partition of X;
2. there exists $T > 0$ such that $\theta(T)$ is the trivial partition;
3. for any $t \geq 0$, there exists $\epsilon > 0$ such that $\theta(t) = \theta(t + \epsilon)$.

Example 2.26. Consider $X = \{a, b, c\}$. Define $\theta : [0, \infty) \to Part(X)$ by

$$\theta(t) = \begin{cases} \{\{a\}, \{b\}, \{c\}\} & \text{if } 0 \leq t < 1 \\ \{\{a, b\}, \{c\}\} & \text{if } 1 \leq t < 2 \\ \{\{a, b, c\}\} & \text{if } t \geq 2 \end{cases}$$

Notice that the strict inequalities cannot be replaced with non-strict inequalities, as this would violate the last condition.

Theorem 2.27 (Finite dendrograms and ultrametrics are equivalent)**.** *Let X be a finite set. Let $\mathcal{D}(X)$ be the set of dendrograms on X and $\mathcal{U}(X)$ be the set of ultrametrics on X. There are two maps between $\mathcal{D}(X)$ and $\mathcal{U}(X)$ as follows:*

1. *(From dendrogram to ultrametric) Given a dendrogram θ on X, define*

   $$u(x, y) := \inf\{t \geq 0 : \ x, y \textit{ belong to the same block in } \theta(t)\} \quad (x, y \in X).$$

   Then u is an ultrametric on X.

2. *(From ultrametric to dendrogram) Given an ultrametric u on X set, we define an equivalence relation $\sim_t$ for $t \geq 0$,*

   $$x \sim_t y \iff u(x, y) \leq t.$$

   Then we define θ by $\theta(t) := \{[x]_t : x \in X\}$ where $[x]_t$ is the equivalence class of x under $\sim_t$. This gives a dendrogram on X.

The above two maps, denoted by $\mathcal{D} : \mathcal{D}(X) \to \mathcal{U}(X)$ and $\mathcal{U} : \mathcal{U}(X) \to \mathcal{D}(X)$ respectively, are inverses of each other.

Example 2.28. Consider the dendrogram θ on $X = \{a, b, c\}$ defined in the previous example. The corresponding ultrametric u is given by

$$u(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } \{x, y\} = \{a, b\}, \\ 2 & \text{if } \{x, y\} = \{a, c\} \text{ or } \{b, c\}. \end{cases}$$

Conversely, starting from this ultrametric u, we can recover the original dendrogram θ.

3 Topological spaces and continuous functions

This section introduces the notion of a topological space, which abstracts the concept of "openness" captured by metric spaces.

Definition 3.1 (Topological Space)**.** A *topology* on a set X is a family τ of subsets of X (called *open sets*) satisfying:

1. Both $\varnothing$ and X belong to τ;
2. Arbitrary unions of sets in τ belong to τ: if $\{U_i\}_{i \in I} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$;
3. Finite intersections of sets in τ belong to τ: if $\{V_j\}_{j=1}^n \subseteq \tau$, then $\bigcap_{j=1}^n V_j \in \tau$.

The pair (X, τ) is called a *topological space*. In many cases, we will omit the mention of the topology and simply refer to the space X.

Definition 3.2. Let (X, τ) be a topological space. A subset $A \subseteq X$ is called *closed* if its complement $X \setminus A$ is open, i.e., $X \setminus A \in \tau$.

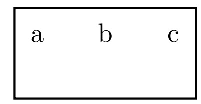
Proposition 3.3. *1. X and $\varnothing$ are closed.*

2. Any finite union of closed sets is closed.

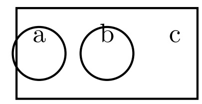
3. Any arbitrary intersection of closed sets is closed.

Remark 3.4. A topology, i.e., the collection of open sets, can be equivalently described by the collection of closed sets.

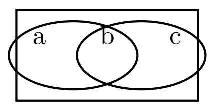
Example 3.5. Which of the following collections of subsets of $X = \{a, b, c\}$ are topologies?



Yes: $\{\varnothing, X\}$ (trivial topology)



No



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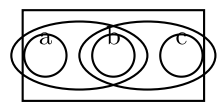
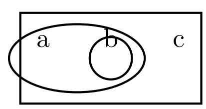
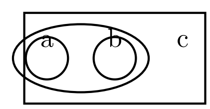


Yes



Yes



Yes

Example 3.6. Every metric space is a topological space with the open sets defined previously.

As a special example, for $\mathbb{R}$, every open ball is a bounded open interval (a, b). Hence, every open set in $\mathbb{R}$ is a union of disjoint open intervals (including unbounded intervals).

Let (X, τ) be a topological space. If there exists a metric d on X such that open sets for d coincide with τ, we say (X, τ) is *metrizable*.

Example 3.7. Let X be any set. Let $\tau := P(X)$. Then, τ is a topology on X and it is called the *discrete topology*. This topology is induced by the discrete metric $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$.

Example 3.8. Let X be any set. Let $\tau := \{\emptyset, X\}$. This is the *trivial topology*. Is this metrizable?

Lemma 3.9. *For a metric space (X, d), any point $x \in X$ forms a closed set, i.e., $\{x\}$ is closed.*

Proof. We show that $X \setminus \{x\}$ is open. For any $y \in X \setminus \{x\}$, let $r = d(x, y)/2 > 0$. Then, the open ball $B(y, r) \subseteq X \setminus \{x\}$. Hence, $X \setminus \{x\}$ is open. $\square$

Now, for any set X with more than two elements, the trivial topology is not metrizable since singletons are not closed.

Example 3.10. Let X be any set. Let $\tau := \{O \subset X \mid O = \emptyset \text{ or } X \setminus O \text{ is finite}\}$. Then τ is a topology on X called the *finite complement topology*.

If $X = \mathbb{R}$, a nonempty open set is $\mathbb{R}$ with at most finitely many points removed. For example:

Nonempty open set: $\mathbb{R} \setminus \{x_1, \dots, x_n\}$

Proof. Now we check that the finite complement topology is indeed a topology.

- The empty set and the whole space X are in τ.
- The intersection of finitely many open sets is open: If $U_1, \dots, U_n \in \tau$ then each $X \setminus U_i$ is finite. Hence

  $$X \setminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X \setminus U_i)$$

  is a finite union of finite sets, hence finite. Therefore $\bigcap_{i=1}^{n} U_i \in \tau$.
- The union of any collection of open sets is open: If $\{U_i\}_{i\in I} \subseteq \tau$, then $X \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I}(X \setminus U_i) \subset X \setminus U_i$ is finite. Thus, $\bigcup_{i\in I} U_i \in \tau$.

$\square$

The finite complement topology coincides with the trivial topology if X is finite and hence is metrizable. If X is infinite, the finite complement topology is not metrizable. We will need more machinery to show this.

Definition 3.11 (Comparing topologies)**.** Let τ and τ' be topologies on the same set X. If $\tau \subseteq \tau'$ we say τ is *coarser* (or *weaker*) than τ', and τ' is *finer* (or *stronger*) than τ.

Remark 3.12. The discrete topology $P(X)$ is the finest topology on X, while the trivial topology $\{\varnothing, X\}$ is the coarsest.

3.1 Basis

Recall that in a metric space, any open set can be written as a union of open balls. So the set of open balls gives rise to a simple characterization of the metric topology. We can use this idea to define a basis for a topology.

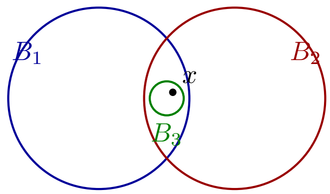
Definition 3.13 (Basis)**.** A *basis* $\mathcal{B}$ for a topology on a set X is a nonempty family of subsets of X such that:

1. For every $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$.
2. If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

The topology generated by $\mathcal{B}$ is: any $U \subset X$ is open if for every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Equivalently,

$$\tau_{\mathcal{B}} := \Big\{ \bigcup_{i \in I} B_i : B_i \in \mathcal{B} \Big\}.$$

We say "$\mathcal{B}$ is a basis for $\tau_{\mathcal{B}}$".



If $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Example 3.14. Let X be a metric space. Then, the collection of open balls forms a basis for the topology on X.

Example 3.15. The collection of one point sets $\{\{x\} : x \in X\}$ is a basis for the discrete topology on X.

Example 3.16. Let τ be a topology on X. Then, $\mathcal{B} := \tau$ is a basis for τ. This is a rather trivial case.

Proposition 3.17. *The topology generated by a basis is indeed a topology.*

Proof.

- $\varnothing$ and X are in $\tau_{\mathcal{B}}$: $\varnothing = \bigcup_{i \in \varnothing} B_i$ and $X = \bigcup_{x \in X} B_x$ where $B_x \in \mathcal{B}$ contains x.
- Arbitrary unions of sets in $\tau_{\mathcal{B}}$ belong to $\tau_{\mathcal{B}}$: Consider $\{U_i\}_{i \in I} \subseteq \tau_{\mathcal{B}}$. Then, if $x \in \bigcup_{i \in I} U_i$, there exists $i_0 \in I$ such that $x \in U_{i_0}$. Since U_{i_0} is a union of basis elements, we can find $B \in \mathcal{B}$ with $x \in B \subseteq U_{i_0}$. Thus, $x \in B \subseteq \bigcup_{i \in I} U_i$. Hence, $\bigcup_{i \in I} U_i \in \tau_{\mathcal{B}}$.
- Now for $U_1, \dots, U_n \in \tau_{\mathcal{B}}$, let $x \in U_1 \cap \cdots \cap U_n$. We claim that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_1 \cap \cdots \cap U_n$. We prove by induction on n. The case $n = 1$ is trivial. Assume the claim holds for $n = k$. Now consider $n = k + 1$. By the induction hypothesis, there exists $B' \in \mathcal{B}$ such that $x \in B' \subseteq U_1 \cap \cdots \cap U_k$. Since U_{k+1} is open, there exists $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq U_{k+1}$. Now, by the definition of basis, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$. Hence, $x \in B \subseteq B' \cap B'' \subseteq (U_1 \cap \cdots \cap U_k) \cap U_{k+1} = U_1 \cap \cdots \cap U_{k+1}$. This proves the claim for $n = k + 1$ and hence for all n.

$\square$

Proposition 3.18. *Let (X, τ) be a topological space and let $\mathcal{B} \subset \tau$. Assume that for any $x \in U$ where $U \in \tau$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$. Then $\mathcal{B}$ is a basis for τ.*

Proof. By hypothesis, for any open $U \in \tau$ and any $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$.

(1) Take $U = X \in \tau$. Then for every $x \in X$ there is $B \in \mathcal{B}$ with $x \in B \subseteq X$, verifying the first basis axiom.

(2) If $x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B} \subseteq \tau$, then $U := B_1 \cap B_2 \in \tau$. Applying the hypothesis to U and x yields $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$, which is the second basis axiom.

Thus $\mathcal{B}$ is a basis. Let $\tau_{\mathcal{B}}$ be the topology it generates. Since $\mathcal{B} \subseteq \tau$ and τ is closed under unions, $\tau_{\mathcal{B}} \subseteq \tau$. Conversely, for any $U \in \tau$,

$$U = \bigcup_{x \in U} B_x$$

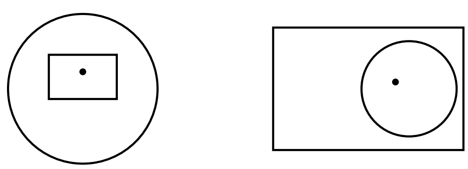
where each $B_x \in \mathcal{B}$ satisfies $x \in B_x \subseteq U$ by the hypothesis. Hence $U \in \tau_{\mathcal{B}}$, so $\tau \subseteq \tau_{\mathcal{B}}$. Therefore $\tau = \tau_{\mathcal{B}}$, and $\mathcal{B}$ is a basis for τ. □

Proposition 3.19 (Comparing topologies via bases)**.** *Let $\mathcal{B}_1, \mathcal{B}_2$ be bases on X generating τ_1, τ_2. Then $\tau_1 \subseteq \tau_2$ iff for every $x \in X$ and $B_1 \in \mathcal{B}_1$ with $x \in B_1$ there exists $B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_1$.*

Proof. ($\Rightarrow$) If $B_1 \in \mathcal{B}_1$, then $B_1 \in \tau_1 \subseteq \tau_2$ and so B_1 is a union of $\mathcal{B}_2$–elements. Pick one containing x inside B_1.

($\Leftarrow$) Let $U \in \tau_1$; write $U = \bigcup_i B_i$ with $B_i \in \mathcal{B}_1$. For $x \in U$ choose i with $x \in B_i$ and then $B_2 \in \mathcal{B}_2$ with $x \in B_2 \subseteq B_i \subseteq U$. Hence U is open in τ_2 by the basis criterion. □

Example 3.20. Let $X = \mathbb{R}^2$. Let $\mathcal{B} = \{\text{open balls in } \mathbb{R}^2\}$ and let $\mathcal{B}' = \{\text{axis-aligned open rectangles in } \mathbb{R}^2\}$. Then they generate the same topology.



3.1.1 Subbasis

Now consider the case where we are just give a set of subsets, can we still generate a topology?

Definition 3.21 (Subbasis)**.** A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets whose union is X. The topology *generated* by $\mathcal{S}$ is the collection of all unions of finite intersections of members of $\mathcal{S}$:

$$\tau_{\mathcal{S}} = \Big\{ \bigcup_{\alpha} \bigcap_{k=1}^{n_\alpha} S_{\alpha,k} : S_{\alpha,k} \in \mathcal{S},\ n_\alpha \geq 0 \Big\}.$$

Example 3.22. $\mathcal{S} = \{\{0,1\},\{0,2\}\}$ is a subbasis on $X = \{0,1,2\}$. It is clear that $\mathcal{S}$ is not a basis. The topology generated by $\mathcal{S}$ is $\tau_{\mathcal{S}} = \{\emptyset, \{0\}, \{0,1\}, \{0,2\}, X\}$ with a basis $\{\{0\},\{0,1\},\{0,2\}\}$.

Theorem 3.23. *$\tau_{\mathcal{S}}$ is a topology on X with the family of all finite intersections of members of $\mathcal{S}$ being a basis.*

Proof. Let $\mathcal{B}$ be the family of all finite intersections of members of $\mathcal{S}$. We first show that $\mathcal{B}$ is a basis.

- For any $x \in X$, since $\bigcup_{S \in \mathcal{S}} S = X$, there exists $S \in \mathcal{S}$ such that $x \in S$. Hence, $x \in S \in \mathcal{B}$.
- For any $B_1 = \bigcap_{i=1}^m S_i$ and $B_2 = \bigcap_{j=1}^n S'_j$ in $\mathcal{B}$, we have that $B_1 \cap B_2 = \bigcap_{i=1}^m S_i \cap \bigcap_{j=1}^n S'_j = \bigcap_{k=1}^{m+n} S_k \in \mathcal{B}$ for some $S_k \in \mathcal{S}$.

Then, it follows from the definition of $\tau_{\mathcal{S}}$ that $\tau_{\mathcal{S}} = \tau_{\mathcal{B}}$. □

Remark 3.24. $\tau_{\mathcal{S}}$ is the smallest topology containing $\mathcal{S}$, i.e., any other topology on X containing $\mathcal{S}$ must also contain $\tau_{\mathcal{S}}$.

Example 3.25. In $\mathbb{R}$ the open intervals (a,b) form a basis. The family $\{(-\infty, a), (a, \infty) : a \in \mathbb{R}\}$ is a subbasis generating the same topology.

3.2 Kuratowski's approach

As we've seen previously that the continuity of a function between metric spaces can be captured completely by closed sets and the notion of closure. This perspective resulted in the following definition of topology which is of course equivalent to the one given above.

Definition 3.26 (Closure Operation)**.** Let X be a set and let

$$\bar{\cdot} : \mathcal{P}(X) \to \mathcal{P}(X)$$

be a map. It is called a *closure operation* if for all $A, B \subset X$:

(i) $\overline{\emptyset} = \emptyset$,

(ii) $A \subset \overline{A}$,

(iii) If $D = \overline{A}$, then $\overline{D} = D$,

(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

We now show that the closure operation and the topology we defined previously give rise to equivalent structures on a set. To this end, we identify the notion of a closed set in a topological space first.

Proposition 3.27. *Given a topological space (X, τ), define*

$$\overline{A} := \bigcap_{\substack{A \subset C \\ C \text{ closed}}} C.$$

Then $\bar{\cdot}$ is a closure operation in the sense of Definition 2.

Proof. (i) Since X is open, $\emptyset$ is closed, so $\overline{\emptyset} = \emptyset$.

(ii) A is contained in every closed superset, so $A \subset \overline{A}$.

(iii) If D is closed and contains A, then taking closure again yields D itself.

(iv) For $A, B \subset X$: $\overline{A} \cup \overline{B}$ is closed and it contains $A \cup B$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Since $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, we conclude that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

□

Proposition 3.28. *Conversely, if $\bar{\cdot}$ is a closure operation as in Definition 2, then the sets*

$$\mathcal{C} := \{\overline{A} : A \subset X\}$$

form the closed sets of a topology in the sense of Definition 1.

Proof. (a) From (i), $\emptyset \in \mathcal{C}$; from (ii), $X = \overline{X} \in \mathcal{C}$.

(b) If $\overline{A}, \overline{B} \in \mathcal{C}$, then by (iii) and (iv), $\overline{A} \cup \overline{B} = \overline{A \cup B} \in \mathcal{C}$. Inductively, one can show that finite unions of closed sets remain closed.

(c) Now consider an arbitrary collection $\{\overline{A_i}\}_{i \in I} \subseteq \mathcal{C}$. We first observe that if $A \subset B$ then $\overline{A} \subset \overline{B}$. This is due to the fact that $\overline{B} = \overline{A \cup B \setminus A} = \overline{A} \cup \overline{(B \setminus A)}$. So for any $i \in I$, we have that $\overline{\bigcap_{i \in I} \overline{A_i}} \subset \overline{\overline{A_i}} = \overline{A_i}$. Therefore, $\bigcap_{i \in I} \overline{A_i} \subseteq \overline{\bigcap_{i \in I} \overline{A_i}} \subseteq \bigcap_{i \in I} \overline{A_i}$. This implies that $\bigcap_{i \in I} \overline{A_i} = \overline{\bigcap_{i \in I} \overline{A_i}} \in \mathcal{C}$.

□

3.2.1 Closure and Interior

Definition 3.29 (Interior)**.** Let X be a topological space and let $A \subseteq X$. The *interior* of A, denoted $\operatorname{int}(A)$ or $\mathring{A}$, is the largest open set contained in A. Equivalently, it is the union of all open sets contained in A:

$$\operatorname{int}(A) = \bigcup \{U \in \tau_X : U \subseteq A\}.$$

Remark 3.30. For any subset $A \subset X$, we have that

$$\operatorname{int}(A) \subseteq A \subseteq \overline{A}.$$

Example 3.31. Let $X = \mathbb{R}$. Then, $\operatorname{int}((0,1]) = (0,1)$ and $\overline{(0,1]} = [0,1]$.

Definition 3.32. Let X be a topological space. For any $x \in X$, we say any open set containing x is a *(open) neighbourhood* of x.

Theorem 3.33. *Let X be a topological space and let $A \subseteq X$. Then,*

- *$x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*
- *$x \in \mathrm{int}(A)$ if and only if there exists a neighbourhood of x contained in A.*

Proof. The second item follows from the definition directly. We prove the first item below.

Let $x \in \overline{A}$. Assume that a neighborhood U of x does not intersect A. Then, $A \subseteq X \setminus U$. Notice that $X \backslash U$ is closed and contains A. Then, by definition of closure, we have $x \in \overline{A} \subseteq X \backslash U$, which is a contradiction.

Now, we assume $x \notin \overline{A}$. Then, there exists a closed set C containing A such that $x \notin C$. In particular, $x \in X \backslash C$. Note that $X \backslash C$ is open and $(X \setminus C) \cap A = \emptyset$. $\square$