3.3 Order topology

Definition 3.34 (Linearly ordered set and intervals)**.** Let $(X, \leq)$ be a linearly ordered (totally ordered) set. For $a, b \in X$ with $a < b$ define

$$(a, b) = \{x \in X : a < x < b\}, \quad [a, b) = \{x \in X : a \leq x < b\},$$

$$(a, b] = \{x \in X : a < x \leq b\}, \quad [a, b] = \{x \in X : a \leq x \leq b\}.$$

Definition 3.35 (Order topology)**.** Let $(X, \leq)$ be a linearly ordered set with at least two elements. Let $\mathcal{B}$ be the family consisting of

- all open intervals (a, b) with $a < b$,
- all $[a_0, b)$ when a_0 is the least element of X and $b > a_0$,
- all $(a, b_0]$ when b_0 is the greatest element of X and $a < b_0$.

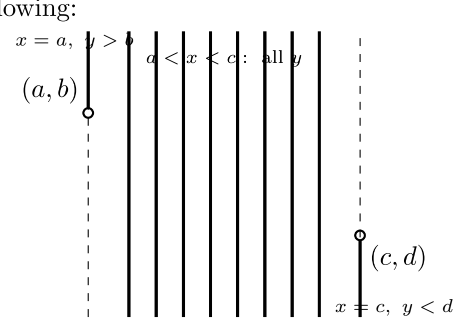
The topology generated by $\mathcal{B}$ is called the *order topology* on X.

Proposition 3.36. *The family $\mathcal{B}$ above is a basis.*

Proof. Nonemptiness is clear. Given $x \in X$, if x is minimal, then x is in $[x, b)$; if x is maximal, then x is in $(a, x]$; otherwise, there must exist a, b such that $x \in (a, b)$. The intersection $B_1 \cap B_2$ with $B_i \in \mathcal{B}$ must be one of the sets in $\mathcal{B}$ or emptyset. $\square$

Example 3.37 (Standard topology on $\mathbb{R}$)**.** With the usual order on $\mathbb{R}$, the sets (a, b) already form $\mathcal{B}$, hence the order topology coincides with the usual metric topology on $\mathbb{R}$.

Example 3.38 (Lexicographic order on $\mathbb{R} \times \mathbb{R}$)**.** Define $(x_1, y_1) < (x_2, y_2)$ iff $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$. The interval $((a, b), (c, d))$ looks like the following:



This is different from the standard topology on $\mathbb{R}^2$. In fact, the set above in the basis is not open in the standard topology. To see this, consider the point $(a, b + 1)$ in the set. Any open ball around it will contain points with $x < a$, which are not in the set.

Example 3.39 (Discrete order on the integers)**.** On $\mathbb{N}$ with the usual order, the order topology is discrete:

$$\{n\} = (n-1, n+1) \quad \text{for } n > 1;$$

and 1 is the least element then $\{1\} = [1, 2)$. Hence every singleton is open.

Example 3.40 (Order topology on $\{1,2\} \times \mathbb{N}$ (lexicographic) is not discrete)**.** Give $X = \{1,2\} \times \mathbb{N}$ the lexicographic order:

$$(i, m) < (j, n) \quad \text{iff} \quad i < j \ \text{ or } \ (i = j \text{ and } m < n).$$

Then $(1,1)$ is the least element. In the order topology, any basis element containing $(2,1)$ is of the form (a, b) (or $[\,(1,1), b)$) with $a < (2,1) < b$, and therefore must contain some $(1, n)$. Hence $\{(2,1)\}$ is not open, so the order topology on X is not the discrete topology.

3.4 Product Topology on $X \times Y$

Definition 3.41. Let (X, τ_X) and (Y, τ_Y) be topological spaces. The *product topology* on the product space $X \times Y$ is the topology generated by the basis consisting of all products of open sets:

$$\mathcal{B}_{X \times Y} = \{U \times V : U \in \tau_X, V \in \tau_Y\}.$$

Check: $\mathcal{B}$ is a basis. Note $X \times Y \in \mathcal{B}$ so every point $(x, y) \in X \times Y$ lies in some basis element.

For $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$, we have $U_1 \cap U_2 \in \tau_X$ and $V_1 \cap V_2 \in \tau_Y$, so

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}.$$

$\square$

Question: is $\mathcal{B}$ a topology? No, the union of basis elements is not necessarily a basis element.

It turns out that smaller bases can also generate the product topology.

Theorem 3.42. *If $\mathcal{B}$ is a basis for τ_X and $\mathcal{B}'$ is a basis for τ_Y, then*

$$\mathcal{B} \times \mathcal{B}' = \{U \times V : U \in \mathcal{B}, V \in \mathcal{B}'\}$$

is a basis for the product topology on $X \times Y$.

Proof. Pick any $U \times V$ from the basis for the product topology. Now, for any $(x, y) \in U \times V$, we have that $x \in U$ and $y \in V$. Since $\mathcal{B}$ and $\mathcal{B}'$ are bases, there exist $U_x \in \mathcal{B}$ and $V_y \in \mathcal{B}'$ such that $x \in U_x \subseteq U$ and $y \in V_y \subseteq V$. Hence, $(x, y) \in U_x \times V_y \subseteq U \times V$. This implies that the topology generated by $\mathcal{B} \times \mathcal{B}'$ is finer than the product topology. On the other hand, since $\mathcal{B} \times \mathcal{B}'$ is a subset of the basis for the product topology, the topology generated by $\mathcal{B} \times \mathcal{B}'$ is coarser than the product topology. Therefore, they must be the same. $\square$

Example 3.43. Let $X = Y = \mathbb{R}$ with the standard topology. Then, $\mathcal{B} = \{(a,b) : a < b\}$ is a basis for $\mathbb{R}$. Hence, $\mathcal{B} \times \mathcal{B} = \{(a,b) \times (c,d) : a < b, c < d\}$ is a basis for the product topology on $\mathbb{R}^2$. Note that $(a,b) \times (c,d)$ is an axis-aligned open rectangle in $\mathbb{R}^2$. Previously, we showed that this generates the same topology as the one generated by open balls in $\mathbb{R}^2$. So the product topology on $\mathbb{R}^2$ is the same as the standard metric topology on $\mathbb{R}^2$.

We can inductively define the product topology on $X_1 \times X_2 \times \cdots \times X_n$ for any finite n. But things will be a bit different for infinite products. We will discuss this later.

Definition 3.44. Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x,y) = x$ and $\pi_2 : X \times Y \to Y$ be defined by $\pi_2(x,y) = y$. The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Note that for any open set $U \subset X$, we have that $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ for any open set $V \subset Y$.

Theorem 3.45. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \in \tau_X\} \cup \{\pi_2^{-1}(V) : V \in \tau_Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Note that for any $U \in \tau_X$ and $V \in \tau_Y$, we have that

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Hence, the basis generated by $\mathcal{S}$ contains all basis elements of the product topology. On the other hand, elements in $\mathcal{S}$ are open in the product topology. Therefore, the topology generated by $\mathcal{S}$ is the same as the product topology. □

3.5 Subspace Topology

Definition 3.46. Suppose (X, τ) is a topological space and let A be a subset of X. The *subspace topology* on A is the collection $\tau_A := \{U \cap A : U \in \tau\}$. Then (A, τ_A) is a topological space.

We now check that τ_A is indeed a topology on A.

- $\varnothing$ and A are in τ_A: $\varnothing = \varnothing \cap A$ and $A = X \cap A$.
- Arbitrary unions of sets in τ_A belong to τ_A: If $\{U_i \cap A\}_{i \in I} \subseteq \tau_A$, then

$$\bigcup_{i \in I} (U_i \cap A) = \Big(\bigcup_{i \in I} U_i\Big) \cap A \in \tau_A.$$

- Finite intersections of sets in τ_A belong to τ_A: If $U_1 \cap A, \ldots, U_n \cap A \in \tau_A$, then

$$(U_1 \cap A) \cap \ldots \cap (U_n \cap A) = (U_1 \cap \ldots \cap U_n) \cap A \in \tau_A.$$

Example 3.47. $[0,1]$ is a subspace of $\mathbb{R}$. Note that $(0.5,1]$ is not open in $\mathbb{R}$ but open in $[0,1]$ with the subspace topology as

$$(0.5,1] = (0.5,2) \cap [0,1]$$

Lemma 3.48. *Let (X,τ) be a topological space and $Y \subseteq X$. If $Y \in \tau$ and $U \in \tau_Y$ (i.e. U is open in the subspace Y), then $U \in \tau$.*

Proof. By definition of the subspace topology, $U \in \tau_Y$ means there exists $V \in \tau$ with $U = V \cap Y$. Since $Y \in \tau$ and τ is closed under finite intersections, $U = V \cap Y$ is open in X. Hence $U \in \tau$. $\square$

Lemma 3.49. *Let (X,τ) be a topological space and let $\mathcal{B}$ be a basis for τ. If $Y \subseteq X$ then*

$$\mathcal{B}_Y := \{\, B \cap Y : B \in \mathcal{B} \,\}$$

is a basis for the subspace topology τ_Y on Y.

Proof. Given any $U \cap Y$ open (U is open in X) in Y and $y \in U \cap Y$, there exists $B \in \mathcal{B}$ such that $y \in B \subseteq U$. Then, $y \in B \cap Y \subseteq U \cap Y$. Thus, $\mathcal{B}_Y$ is a basis for τ_Y. $\square$

Order topology versus subspace topology

Theorem 3.50. *If $A \subseteq X$ and $B \subseteq Y$, then the product topology on $A \times B$ is the same as the subspace topology inherited from $X \times Y$.*

Proof. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be bases for X and Y respectively. Then, $\mathcal{B}_X \times \mathcal{B}_Y := \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$. Hence, the subspace topology of $A \times B$ has a basis

$$\mathcal{B}_{A\times B} = \{(U \times V) \cap (A \times B) : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}.$$

Note that $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$.

Then,

$$\mathcal{B}_{A\times B} = \{(U \cap A) \times (V \cap B) : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\} = \mathcal{B}_A \times \mathcal{B}_B.$$

This finishes the proof. $\square$

Theorem 3.51. *Let (X,d) be a metric space and let $Y \subseteq X$. Consider the sub metric space (Y,d_Y). Then, the topology induced by d_Y is the same as the subspace topology inherited from (X,d).*

Although the subspace topology and the metric topology are compatible, the order topology and the subspace topology are not always compatible. We will see when they agree.

Remark 3.52. The order and subspace topologies on a subset of a linearly ordered set need not agree. For instance, let $Y = [0,1) \cup \{2\} \subset \mathbb{R}$ with the induced order. In the subspace topology, $\{2\}$ is open in Y since $\{2\} = (2-\frac{1}{2}, 2+\frac{1}{2}) \cap Y$. But in the order topology on Y, any basis element containing 2 is of the form $(a,2]_Y = \{y \in Y : a < y \leq 2\}$ where $a \in Y$ (because 2 is the greatest element of Y), so it contains points of Y strictly less than 2; hence $\{2\}$ is not open there.

Definition 3.53 (Convex subset)**.** Let $(X, \leq)$ be a linearly ordered set. A subset $Y \subset X$ is *convex* if for all $a, b \in Y$ with $a < b$ one has $(a,b)_X \subset Y$.

Theorem 3.54. *Let X be a linearly ordered set with the order topology, and let $Y \subset X$ be convex (with the induced order). Then the order topology on Y coincides with the subspace topology inherited from X.*

For a proof, see Munkres, Theorem 16.4.

3.6 Continuous functions

Now we consider the functions between two topological spaces.

We first define the continuity using a local idea adopted from metric spaces. Then, we will see some more global characterizations.

Definition 3.55. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ between topological spaces is called *continuous* at a given point $x \in X$ if for every neighbourhood V of $f(x)$ in Y, there exists a neighbourhood U of x in X such that $f(U) \subseteq V$. We say that f is *continuous* if it is continuous at every point $x \in X$.

Example 3.56. A function $f : X \to Y$ between two metric spaces is continuous in the metric sense if and only if it is continuous in the topological sense.

Theorem 3.57. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. The following are equivalent:*

1. *f is continuous.*
2. *For all open sets H in Y, $f^{-1}(H)$ is open in X.*
3. *For all closed sets C in Y, $f^{-1}(C)$ is closed in X.*
4. *For all $A \subset X$, $f(\overline{A}) \subseteq \overline{f(A)}$ (This can be thought of as the Kuratowski definition of continuity).*

Proof. We are going to use the following observation several times: For any $A \subseteq Y$, we have that

$$X = f^{-1}(A) \cup f^{-1}(Y \setminus A).$$

(1) $\Rightarrow$ (2): Let H be open in Y. If $f^{-1}(H) = \emptyset$, then done. Otherwise, for any $x \in f^{-1}(H)$, H is a neighborhood of $f(x)$. Hence, there exists a neighborhood V of x such that $f(V) \subseteq H$. Thus, $V \subseteq f^{-1}(H)$. Therefore, $f^{-1}(H)$ is open in X.

(2) $\Rightarrow$ (3): If C is closed in Y, then $Y \setminus C$ is open in Y. By (2), $f^{-1}(Y \setminus C)$ is open in X. Then, $f^{-1}(C) = X \setminus f^{-1}(Y \setminus C)$ is closed in X.

(3) $\Rightarrow$ (4): $f^{-1}(\overline{f(A)})$ is closed by (2). Furthermore, $A \subseteq f^{-1}(\overline{f(A)})$ (since $f(A) \subseteq \overline{f(A)}$). This implies that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ and hence $f(\overline{A}) \subseteq \overline{f(A)}$.

(4) $\Rightarrow$ (1): Let $x \in X$ and let V be a neighborhood of $f(x)$ in Y. Set $E := f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Then, we have that $f(\overline{E}) \subset \overline{f(E)} \subseteq Y \setminus V$ (because $Y \setminus V$ is a closed set containing $f(E)$).

Now we have that $x \in X \setminus \overline{E}$, which is a neighborhood of x. Then,

$$f(X \setminus \overline{E}) \subset f(X \setminus E) = f(X \setminus f^{-1}(Y \setminus V)) = f(f^{-1}(V)) \subseteq V.$$

This concludes the proof. $\square$

Remark 3.58. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if and only if the statement (2) holds for all basis elements of Y:

$$U = \cup B_\alpha \quad f^{-1}(U) = \cup f^{-1}(B_\alpha)$$

Remark 3.59. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if and only if the statement (2) holds for all subbasis elements of Y:

$$B = S_1 \cap \cdots \cap S_n \quad f^{-1}(B) = \cap f^{-1}(S_i)$$

Remark 3.60. In some books, they take the (2) as the definition of continuity.

Example 3.61. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant map $f(x) = 0$. Then f is continuous, but the image of the open set $(0, 1)$ is

$$f((0, 1)) = \{0\},$$

which is not open in $\mathbb{R}$. Thus a continuous map need not send open sets to open sets.

Example 3.62. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi(x, y) = x$, which is continuous. Consider

$$C = \{(x, y) \in \mathbb{R}^2 : xy = 1\},$$

the hyperbola. Since C is the zero set of the continuous function $f : (x, y) \mapsto xy - 1$, it is closed in $\mathbb{R}^2$ (i.e., $C = f^{-1}(\{0\})$). But

$$\pi(C) = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R} \setminus \{0\},$$

which is not closed in $\mathbb{R}$. Thus a continuous map need not send closed sets to closed sets.

Example 3.63. Let $X \times Y$ be endowed with the product topology. Then, $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

Note that for any open set U in X, $\pi_X^{-1}(U) = U \times Y$ is open in $X \times Y$. Similarly, for any open set V in Y, $\pi_Y^{-1}(V) = X \times V$ is open in $X \times Y$.

Note also that the product topology $\tau_{X \times Y}$ is the smallest topology on $X \times Y$ such that both π_X and π_Y are continuous: any topology on $X \times Y$ that makes both π_X and π_Y continuous must contain a basis for $\tau_{X \times Y}$: sets of the form $U \times Y$ and $X \times V$ for all open sets U in X and V in Y.