(2) $\Rightarrow$ (3): If C is closed in Y, then $Y \setminus C$ is open in Y. By (2), $f^{-1}(Y \setminus C)$ is open in X. Then, $f^{-1}(C) = X \setminus f^{-1}(Y \setminus C)$ is closed in X.

(3) $\Rightarrow$ (4): $f^{-1}(\overline{f(A)})$ is closed by (2). Furthermore, $A \subseteq f^{-1}(\overline{f(A)})$ (since $f(A) \subseteq \overline{f(A)}$). This implies that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ and hence $f(\overline{A}) \subseteq \overline{f(A)}$.

(4) $\Rightarrow$ (1): Let $x \in X$ and let V be a neighborhood of $f(x)$ in Y. Set $E := f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Then, we have that $f(\overline{E}) \subset \overline{f(E)} \subseteq Y \setminus V$ (because $Y \setminus V$ is a closed set containing $f(E)$).

Now we have that $x \in X \setminus \overline{E}$, which is a neighborhood of x. Then,

$$f(X \setminus \overline{E}) \subset f(X \setminus E) = f(X \setminus f^{-1}(Y \setminus V)) = f(f^{-1}(V)) \subseteq V.$$

This concludes the proof. $\square$

Remark 3.58. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if and only if the statement (2) holds for all basis elements of Y:

$$U = \cup B_\alpha \quad f^{-1}(U) = \cup f^{-1}(B_\alpha)$$

Remark 3.59. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if and only if the statement (2) holds for all subbasis elements of Y:

$$B = S_1 \cap \cdots \cap S_n \quad f^{-1}(B) = \cap f^{-1}(S_i)$$

Remark 3.60. In some books, they take the (2) as the definition of continuity.

Example 3.61. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant map $f(x) = 0$. Then f is continuous, but the image of the open set $(0, 1)$ is

$$f((0, 1)) = \{0\},$$

which is not open in $\mathbb{R}$. Thus a continuous map need not send open sets to open sets.

Example 3.62. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi(x, y) = x$, which is continuous. Consider

$$C = \{(x, y) \in \mathbb{R}^2 : xy = 1\},$$

the hyperbola. Since C is the zero set of the continuous function $f : (x, y) \mapsto xy - 1$, it is closed in $\mathbb{R}^2$ (i.e., $C = f^{-1}(\{0\})$). But

$$\pi(C) = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R} \setminus \{0\},$$

which is not closed in $\mathbb{R}$. Thus a continuous map need not send closed sets to closed sets.

Example 3.63. Let $X \times Y$ be endowed with the product topology. Then, $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

Note that for any open set U in X, $\pi_X^{-1}(U) = U \times Y$ is open in $X \times Y$. Similarly, for any open set V in Y, $\pi_Y^{-1}(V) = X \times V$ is open in $X \times Y$.

Note also that the product topology $\tau_{X \times Y}$ is the smallest topology on $X \times Y$ such that both π_X and π_Y are continuous: any topology on $X \times Y$ that makes both π_X and π_Y continuous must contain a basis for $\tau_{X \times Y}$: sets of the form $U \times Y$ and $X \times V$ for all open sets U in X and V in Y.

3.6.1 Constructing continuous functions

Theorem 3.64. *1. (Constant function) If $f : X \to Y$ maps all of X to a single point $y_0 \in Y$, then f is continuous.*

2. *(Inclusion) If A is a subspace of X, the inclusion map $i : A \to X$ defined by $i(a) = a$ is continuous.*

3. *(Composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composition $g \circ f : X \to Z$ is continuous.*

4. *(Restricting the domain) If $f : X \to Y$ is continuous and $A \subseteq X$, then the restriction $f|_A : A \to Y$ defined by $f|_A(a) = f(a)$ is continuous.*

5. *(Restricting or expanding the range) Let $f : X \to Y$ be continuous. If $f(X) \subset Z \subset Y$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If $Y \subset W$, then the function $h : i_{Y \to W} \circ f : X \to W$ is continuous.*

6. *(Local formulation) $f : X \to Y$ is continuous if $X = \cup U_\alpha$ for a family of open sets U_α and each restriction $f|_{U_\alpha}$ is continuous.*

7. *(Pasting lemma) If $X = A \cup B$ with A, B closed, and $f : A \to Y$, $g : B \to Y$ are continuous and agree on $A \cap B$ (i.e., $f(x) = g(x)$ for all $x \in A \cap B$), then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$$

is continuous.

Proof. 1. For any open set U in Y, if $y_0 \in U$, then $f^{-1}(U) = X$; otherwise, $f^{-1}(U) = \emptyset$. In both cases, $f^{-1}(U)$ is open in X. Hence, f is continuous.

2. For any open set U in X, $i^{-1}(U) = U \cap A$ is open in A. Hence, i is continuous.

3. For any open set W in Z, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since g is continuous, $g^{-1}(W)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in X. Hence, $g \circ f$ is continuous.

4. For any open set U in Y, $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$ is open in A. Hence, $f|_A$ is continuous.

5. For any open set V in Z, then there exists U open in Y such that $V = U \cap Z$. Hence, $g^{-1}(V) = f^{-1}(V) = f^{-1}(U \cap Z)$. Since $f(X) \subset Z$, we have that $f^{-1}(U \cap Z) = f^{-1}(U)$ is open in X. Hence, g is continuous. The continuity of h follows from 2 and 3.

6. For any open set U in Y, we first note that
$$f^{-1}(U) \cap U_\alpha = (f|_{U_\alpha})^{-1}(U) = \{x \in U_\alpha : f(x) \in U\}.$$
Since $f|_{U_\alpha}$ is continuous, $(f|_{U_\alpha})^{-1}(U)$ is open in U_α. Hence,
$$f^{-1}(U) = \bigcup (f^{-1}(U) \cap U_\alpha) = \bigcup (f|_{U_\alpha})^{-1}(U)$$
is open in X. This shows that f is continuous.

7. For any closed set C in Y, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since A and B are closed, $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X (see the later lemma for this). Hence, $h^{-1}(C)$ is closed in X. This shows that h is continuous.

Lemma 3.65. *Let $A \subset X$ and let $E \subset A$. Then, E is closed in A if and only if there exists a closed set F in X such that $E = F \cap A$.*

Proof. Suppose E is closed in A. Then $A \setminus E$ is open in A, so there exists an open set U in X such that $A \setminus E = U \cap A$. Let $F = X \setminus U$, which is closed in X. Then
$$F \cap A = (X \setminus U) \cap A = A \setminus (U \cap A) = A \setminus (A \setminus E) = E.$$
Conversely, if $E = F \cap A$ for some closed set F in X, then
$$A \setminus E = A \setminus (F \cap A) = A \cap (X \setminus F),$$
where $X \setminus F$ is open in X. Hence, $A \setminus E$ is open in A, so E is closed in A. □

Corollary 3.66. *Let $A \subset X$ and let $E \subset A$. If E is closed in A then E is closed in X.*

Proof. Since E is closed in A, there exists a closed set F in X such that $E = F \cap A$. Since $A \subset X$ is closed, we have that E is closed in X □

□

Remark 3.67. 7 is the closed set and finite version of 6. See your homework for the infinite version of 7.

Theorem 3.68. *Let $f : A \to X \times Y$ be given by $f(a) = (f_X(a), f_Y(a))$. Then f is continuous if and only if f_X and f_Y are continuous.*

Proof. Recall the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$. Note that $f_X = \pi_X \circ f$ and $f_Y = \pi_Y \circ f$.

$\Rightarrow$: If f is continuous, then f_X and f_Y are continuous by the composition of continuous functions.

$\Leftarrow$: For $U \times V$ a basis element in $X \times Y$, we have that
$$f^{-1}(U \times V) = f^{-1}(\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)) = f^{-1}(\pi_X^{-1}(U)) \cap f^{-1}(\pi_Y^{-1}(V)) = f_X^{-1}(U) \cap f_Y^{-1}(V).$$
Since f_X and f_Y are continuous, $f^{-1}(U \times V)$ is open in A. This shows that f is continuous. □

3.6.2 Homeomorphism

Definition 3.69. A function $f : X \to Y$ between topological spaces is a *homeomorphism* if it is a bijection and both f and f^{-1} are continuous.

If f is not surjective, we let $Z = f(X)$. If the restriction $f' : X \to Z$ is a homeomorphism, then we say that f is a topological embedding of X into Y.

Homeomorphisms preserve topological properties, meaning that if X and Y are homeomorphic, they share all topological characteristics. More precisely, we have the following theorem.

Theorem 3.70. *Let $f : X \to Y$ be a homeomorphism. Then, f induces a map $f_* : \tau_X \to \tau_Y$ defined by $f_*(U) = f(U)$ for all $U \in \tau_X$. Furthermore, f_* is a bijection.*

Proof. For any $U \in \tau_X$, $f(U) = (f^{-1})^{-1}(U)$. Since f^{-1} is continuous, $f(U)$ is open in Y. This shows that f_* is well-defined.

Next, we show that f_* is injective. If $f_*(U_1) = f_*(U_2)$ for some $U_1, U_2 \in \tau_X$, then $f(U_1) = f(U_2)$. Since f is bijective, we have that $U_1 = U_2$. This shows that f_* is injective.

Finally, we show that f_* is surjective. For any $V \in \tau_Y$, since f^{-1} is continuous, we have that $f^{-1}(V)$ is open in X. Then,

$$f_*(f^{-1}(V)) = f(f^{-1}(V)) = V.$$

This shows that f_* is surjective.

Therefore, we conclude that $f_* : \tau_X \to \tau_Y$ is a bijection. $\square$

Example 3.71. Suppose you have a subspace in some ambient Euclidean space. You can stretch and bend it without tearing or gluing. This is a homeomorphism. For example, a coffee cup is homeomorphic to a donut (solid torus).

Example 3.72. The circle and the trefoil knot are homeomorphic. But we can't deform one to the other in 3D space without cutting.

Example 3.73. The map $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \frac{x}{1-x^2}$ is a homeomorphism with inverse $f^{-1} : \mathbb{R} \to (-1, 1)$ given by $f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}$. So homeomorphisms need not preserve boundedness.

Example 3.74 (Continuous bijection not a homeomorphism)**.** Let

$$X = [0, 1)$$

with the usual topology from $\mathbb{R}$, and

$$Y = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

with the subspace topology from $\mathbb{R}^2$.

Define

$$f : X \to Y, \quad f(t) = (\cos 2\pi t, \sin 2\pi t).$$

Then f is a continuous bijection. However, f^{-1} is not continuous: around $(1,0) \in S^1$, neighborhoods correspond to t near 0 *and* near 1, which are far apart in $[0,1)$, so f^{-1} fails continuity there.
This shows that a continuous bijection need not be a homeomorphism when the domain is not compact.

Example 3.75. Is $(0,1)$ homeomorphic to $[0,1)$?

Lemma 3.76. *Let $f : X \to Y$ be a homeomorphism. Let $x_0 \in X$. Then, the map $g : X\backslash\{x_0\} \to Y\backslash\{f(x_0)\}$ defined by $g(x) = f(x)$ is also a homeomorphism.*

Definition 3.77. Let X be a topological space. We say X is connected if it cannot be represented as the union of two disjoint non-empty open sets.

Remark 3.78. If X is connected and X is homeomorphic to Y, then Y is also connected. This holds obviously as the homeomorphism gives rise to a bijection between the topologies.

Now we are ready to answer the previous question.

Theorem 3.79. $(0,1)$ *is not homeomorphic to* $[0,1)$.

Proof. Suppose there exists a homeomorphism $f : [0,1) \to (0,1)$. Pick any $x_0 = 0$ and consider the map $g : (0,1) \to (0,1)\setminus\{f(x_0)\}$ defined by $g(x) = f(x)$. By the previous lemma, g is a homeomorphism. Note that $(0,1) \setminus \{f(x_0)\}$ is not connected regardless the value of $f(x_0)$. However, $(0,1) = [0,1) \setminus \{x_0\}$ is connected. This is a contradiction. $\square$

Example 3.80. Similarly, we can prove that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.

How about $\mathbb{R}^2$ vs $\mathbb{R}^3$? Need algebraic topology. We will have a taste of it later in the course.

3.7 Product Topology

Recall the Cartesian product of a family of sets $\{X_i\}_{i\in I}$ is the set of functions:

$$\prod_{i\in I} X_i := \{x : I \to \cup_{i\in I} X_i | \text{ for all } i \in I, x(i) \in X_i\}.$$

Let's mimic the case when I is a two-point set to identify the following definition of topology.

Definition 3.81 (Box topology (less important))**.** For spaces (X_i, τ_i), $i \in I$, the *box topology* on $\prod_{i\in I} X_i$ is generated by the basis

$$\mathcal{B}_{\text{box}} = \Big\{ \prod_{i\in I} U_i : U_i \in \tau_i \text{ for every } i \Big\}$$

Let's first verify that $\mathcal{B}_{\text{box}}$ is indeed a basis. For any $x \in \prod_{i\in I} X_i$, we have that $x \in \prod_{i\in I} X_i = \prod_{i\in I} X_i$. Furthermore, for any two basis elements $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$, we have that

$$x \in \prod_{i\in I}(U_i \cap V_i) = \prod_{i\in I} U_i \cap \prod_{i\in I} V_i.$$

Since U_i and V_i are open in X_i, $U_i \cap V_i$ is open in X_i. This shows that $\mathcal{B}_{\text{box}}$ is indeed a basis.

This basis seems reasonable right? But this topology is too fine and makes many things not work.

Example 3.82. Let $X = \mathbb{R}^{\mathbb{N}} = \prod_{n\geq 1} \mathbb{R}$. Then, the "diagonal" map $f : \mathbb{R} \to X$ defined by $f(t) = (t, t, t, \ldots)$ is not continuous when X is endowed with the box topology. Consider the open set $(-1, 1)\times(-1/2, 1/2)\times(-1/3, 1/3)\times\cdots$. Then,

$$\begin{aligned}
&f^{-1}((-1,1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots) \\
&= f^{-1}((-1,1) \times \mathbb{R} \times \cdots \cap \mathbb{R} \times (-1/2, 1/2) \times \mathbb{R} \times \cdots \cap \cdots) \\
&= \bigcap_{n\geq 1} f^{-1}(\mathbb{R} \times \cdots \times (-1/n, 1/n) \times \cdots) \\
&= \bigcap_{n\geq 1} (-1/n, 1/n) = \{0\}.
\end{aligned}$$

Then $f^{-1}((-1,1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \cdots)$ is not open in $\mathbb{R}$. Hence, f is not continuous.

This example actually gives us a hint how we can make things right. The problem above is that when considering the arbitrary intersection of open sets, we don't get an open set. SO, we want things to be finite in some sense. This leads to the following definition.

Definition 3.83 (Product topology)**.** For spaces (X_i, τ_i), $i \in I$, the *product* $X = \prod_{i\in I} X_i$ carries the product topology generated by the basis

$$\mathcal{B}_{\text{prod}} = \Big\{ \prod_{i\in I} U_i : U_i \in \tau_i \text{ and } U_i = X_i \text{ for all but finitely many } i \Big\}.$$

Now we verify that $\mathcal{B}_{\text{prod}}$ is indeed a basis. For any $x \in \prod_{i\in I} X_i$, we have that $x \in \prod_{i\in I} X_i = \prod_{i\in I} X_i$. Furthermore, for any two basis elements $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$, we have that

$$\prod_{i\in I}(U_i \cap V_i) = \prod_{i\in I} U_i \cap \prod_{i\in I} V_i.$$

Since U_i and V_i are open in X_i, $U_i \cap V_i$ is open in X_i. Furthermore, since $U_i = X_i$ for all but finitely many i and $V_i = X_i$ for all but finitely many i, we have that $U_i \cap V_i = X_i$ for all but finitely many i. This shows that $\mathcal{B}_{\text{prod}}$ is indeed a basis.

Remark 3.84. Since $\mathcal{B}_{\text{prod}} \subseteq \mathcal{B}_{\text{box}}$, the product topology is coarser than the box topology. If the index set I is finite, then the box topology and the product topology coincide.

Remark 3.85. The identity map $(\prod_{i\in I} X_i, \tau_{\text{box}}) \to (\prod_{i\in I} X_i, \tau_{\text{prod}})$ is continuous; the inverse need not be when I is infinite.

Example 3.86. Now we revisit this example where $X = \mathbb{R}^{\mathbb{N}} = \prod_{n\geq 1} \mathbb{R}$. Then, the "diagonal" map $f : \mathbb{R} \to X$ defined by $f(t) = (t, t, t, \ldots)$ is continuous when X is endowed with the product topology. Consider any basis element $U = \prod_{n\geq 1} U_n$ in X. Then, there exists $N > 0$ such that for all $n > N$, $U_n = \mathbb{R}$. Then,

$$\begin{aligned} f^{-1}(\prod_{n\geq 1} U_n) &= f^{-1}(\prod_{n=1}^{N} U_n \times \prod_{n>N} \mathbb{R}) \\ &= f^{-1}(\bigcap_{n=1}^{N} \mathbb{R} \times \cdots \times U_n \times \mathbb{R} \times \cdots) \\ &= \bigcap_{n=1}^{N} f^{-1}(\mathbb{R} \times \cdots \times U_n \times \mathbb{R} \times \cdots) \\ &= \bigcap_{n=1}^{N} U_n. \end{aligned}$$

Since (U_n) is open in $\mathbb{R}$ for all n, $f^{-1}(\prod_{n\geq 1} U_n)$ is open in $\mathbb{R}$. Hence, f is continuous.

Example 3.87. Let $X = \mathbb{R}^{\mathbb{N}} = \prod_{n\geq 1} \mathbb{R}$. Then $(0,1)^{\mathbb{N}}$ is box open but not product open (it contains no basis element).

The product topology has a simple subbasis.

Theorem 3.88. *The product topology on $X = \prod_{i\in I} X_i$ is generated by the subbasis*

$$\mathcal{S} = \{\pi_i^{-1}(U_i) : i \in I, U_i \text{ open in } X_i\}$$

where $\pi_i : X \to X_i$ is the projection onto the ith factor.

Proof. Note that for any $i \in I$ and any open set U_i in X_i, $\pi_i^{-1}(U_i)$ is of the form

$$\pi_i^{-1}(U_i) = \prod_{j\in I} V_j$$

where $V_j = U_i$ if $j = i$ and $V_j = X_j$ otherwise.

So, for any finite family of such sets $\pi_{i_1}^{-1}(U_{i_1}), \ldots, \pi_{i_n}^{-1}(U_{i_n})$, we have that

$$\bigcap_{k=1}^{n} \pi_{i_k}^{-1}(U_{i_k}) = \prod_{j\in I} W_j$$

where $W_j = U_{i_k}$ if $j = i_k$ for some $k \in \{1, \dots, n\}$ and $W_j = X_j$ otherwise. This is a basis element in the product topology.

Conversely, it is easy to see that any basis element $\prod_{i\in I} U_i$ in the product topology can be written as a finite intersection of sets in $\mathcal{S}$ as above. $\square$

Here is a one great example showing why people care about the product topology.

Definition 3.89. Let $\pi_i : \prod_{j\in I} X_j \to X_i$ be the projection onto the ith factor, i.e., $\pi_i(x) = x(i)$.

Theorem 3.90 (Minimality)**.** *Let τ_Π be the product topology on $X = \prod_{i\in I} X_i$. It is the smallest topology on X making every projection $\pi_i : X \to X_i$ continuous: if τ is any topology on X with all π_i continuous, then $\tau_\Pi \subseteq \tau$.*

Proof. For any U_i open in X_i, we have that $\pi_i^{-1}(U_i)$ must be contained in τ. So $\tau \supset \mathcal{S}$ and hence $\tau \supset \tau_\Pi$. $\square$

Theorem 3.91. *Consider any function $f : Y \to \prod_{j\in I} X_j$. Then f is continuous iff each coordinate projection $f_i := \pi_i \circ f : Y \to X_i$ is continuous.*

Proof. Note each projection π_i is continuous by the theorem above.

$\Rightarrow$: If f is continuous, then $f_i = \pi_i \circ f$ is continuous by the composition of continuous functions.

$\Leftarrow$: For any subbasis element $\pi_i^{-1}(U_i)$ where U_i is open in X_i, we have that $f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i) = f_i^{-1}(U_i)$ is open in Y since f_i is continuous. This proves that f is continuous. $\square$

Remark 3.92. The $\Leftarrow$ direction needs not be true if we consider the box topology instead of the product topology. The same diagonal map $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ defined by $f(t) = (t, t, t, \dots)$ is a counterexample: every $f_i(t) = t$ is continuous but f is not continuous when $\mathbb{R}^{\mathbb{N}}$ is endowed with the box topology.

Now let's talk about the product of metric spaces. We have one example in homework 3 but we will give a more general construction here.

Lemma 3.93. *Let (X, d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by*

$$d'(x, y) = \min\{d(x, y), 1\}.$$

Then, d' is a metric on X and the topology induced by d' is the same as the topology induced by d.

Proof. It is easy to check that d' is a metric.

Since $d'(x, y) \leq d(x, y)$, then $\tau_{d'} \subset \tau_d$.

Now, we show that $\tau_d \subset \tau_{d'}$. For any $x \in X$ and any $r > 0$, if $r \leq 1$, then $B_d(x, r) = B_{d'}(x, r)$; if $r > 1$, then $B_d(x, r) \supset B_{d'}(x, 1)$. This shows that $\tau_d \subset \tau_{d'}$. $\square$

Theorem 3.94. *Let $\{(X_i, d_i)\}_{i\in\mathbb{N}}$ be a family of metric spaces. Let $X = \prod_{i\in\mathbb{N}} X_i$. For any $i \in \mathbb{N}$, we define $d_i' : X \times X \to \mathbb{R}$ by*

$$d_i'(x, y) = \min\{d_i(x(i), y(i)), 1\}.$$

Then, the function $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \sum_{i\in\mathbb{N}} \frac{d_i'(x, y)}{2^i}$$

is a metric on X and the topology induced by d is the product topology.

The proof is going to be almost identical to problem 3 in homework 3. So we omit it here.

Example 3.95. $\mathbb{R}^{\mathbb{N}}$ with the box topology is not metrizable. Let $A = \{(x_1, \ldots)|x_i > 0\}$ and let $\mathbf{0} = (0, 0, \ldots)$. Then, $\mathbf{0} \in \overline{A}$: for any basis element $U = (a_1, b_1) \times (a_2, b_2) \times \cdots$ containing $\mathbf{0}$, we have that $a_i < 0 < b_i$ for all i. So pick any point $(x_1, x_2, \ldots)$ with $x_i \in (0, b_i)$ for all i, then $(x_1, x_2, \ldots) \in A \cap U$. So, $\overline{A} \cap U \neq \emptyset$.

Assume that $\mathbb{R}^{\mathbb{N}}$ with the box topology is metrizable with a compatible metric d. Let $\mathbf{a}_n = (x_{1n}, x_{2n}, \ldots)$ be a sequence in A converging to $\mathbf{0}$. Then, construct $B = \{(-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots\}$. Then, $\mathbf{0} \in B$ but $\mathbf{a}_n \notin B$ for all $n \in \mathbb{N}$. So there exists a ball $B_r(\mathbf{0}) \subseteq B$ for some $r > 0$ containing no point in the sequence $\mathbf{a}_n$. This is a contradiction. So $\mathbb{R}^{\mathbb{N}}$ with the box topology is not metrizable.

3.8 Quotient spaces

We have seen how to construct topologies on various sets. One way to create a new space from old ones is via gluing. More precisely, let X be a topological space and let X^* be a partition of X (in other words, an equivalence relation $\sim$ on X so that $X^* = X/\sim$).

Example 3.96. One can obtain a torus and a sphere via gluing/equivalence relations.

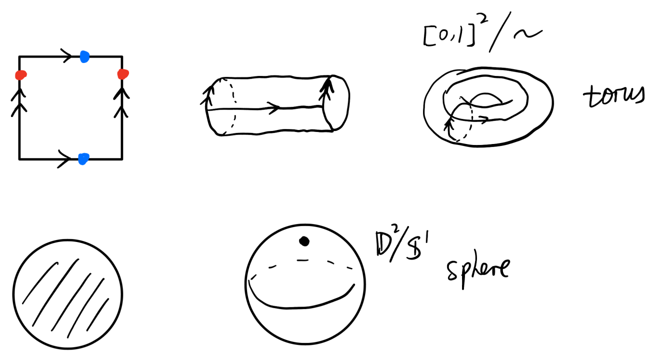

Figure 2: A torus and a sphere, illustrating the concept of quotient spaces.

How can we get a topology on X^* from the topology on X? Recall that the product topology can be characterized via some continuous maps. Maybe we can do something similar here. A very natural map is the quotient map $p : X \to X^*$ sending each point to its equivalence class.

We of course want p to be continuous. That means we want a set of open sets on X^* so that the preimage of each open set is open in X. We want to make this topology as coarse as possible.

Definition 3.97 (Quotient topology)**.** Let $q : X \to Y$ be a surjective map. The *quotient topology* on Y is defined by

$$\tau_q := \{V \subseteq Y \mid q^{-1}(V) \text{ is open in } X\}$$

When endowed with the quotient topology, we call Y a *quotient space* of X and q a *quotient map.*

Check: this is indeed a topology. $q^{-1}(Y) = X$ and $q^{-1}(\emptyset) = \emptyset$. So $Y, \emptyset \in \tau_q$.

$q^{-1}(\bigcup_\alpha V_\alpha) = \bigcup_\alpha q^{-1}(V_\alpha)$, so $\bigcup_\alpha V_\alpha \in \tau_q$.

$q^{-1}(\cap_{i=1}^n V_i) = \cap_{i=1}^n q^{-1}(V_i)$, so $\cap_{i=1}^n V_i \in \tau_q$. $\square$

Remark 3.98. The quotient topology is the finest topology on Y making q continuous: if τ is any topology on Y with q continuous, then $\tau \subseteq \tau_q$.

Definition 3.99 (Quotient space)**.** Let X be a topological space. Let $\sim$ be an equivalence relation on X and let $X^* := X/\sim$ be the set of equivalence classes. The *quotient space* of X by $\sim$ is the set X^* endowed with the quotient topology via the natural projection map $p : X \to X^*$ sending each point to its equivalence class.

Example 3.100 (A 3–point quotient of $\mathbb{R}$)**.** Let $p : \mathbb{R} \to Y = \{-1, 0, 1\}$ be

$$p(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Endow Y with the quotient topology via p. A subset $V \subseteq Y$ is open iff $p^{-1}(V)$ is open in $\mathbb{R}$.

- $p^{-1}(\{-1\}) = (-\infty, 0)$ and $p^{-1}(\{1\}) = (0, \infty)$ are open $\Rightarrow \{-1\}, \{1\}$ are open.
- $p^{-1}(\{0\}) = \{0\}$ is not open $\Rightarrow \{0\}$ not open.
- $p^{-1}(\{-1, 1\}) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ is open $\Rightarrow \{-1, 1\}$ open.
- $p^{-1}(\{-1, 0\}) = (-\infty, 0]$ and $p^{-1}(\{0, 1\}) = [0, \infty)$ are not open.

Thus
$$\tau_Y = \{\varnothing, \{-1\}, \{1\}, \{-1, 1\}, \{-1, 0, 1\}\}.$$

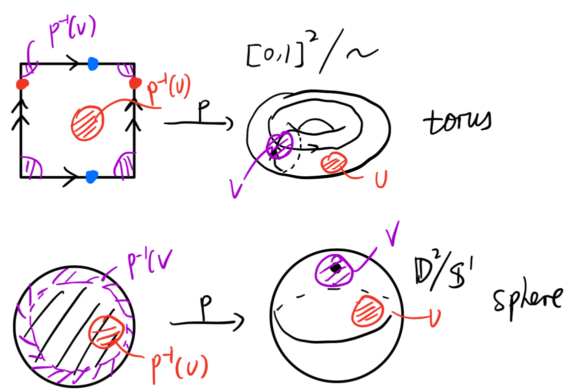

Figure 3: A torus and a sphere, illustrating the concept of quotient spaces.

Example 3.101 (Torus and Sphere)**.** Consider the maps shown in the image (Figure 3). We then can endow the torus and the sphere (viewed as subsets of $\mathbb{R}^3$) with the quotient topologies via these maps. Of course, there is the question whether these quotient topologies coincide with the subspace topologies from $\mathbb{R}^3$. We will answer this question later.