Now let $q : X \to Y$ be surjective. What if we have another topology τ on Y?

Definition 3.102. For two topological spaces X, Y and a surjective map $q : X \to Y$, we say q is a *quotient map* if

$$\text{a subset } U \subseteq Y \text{ is open in } Y \text{ iff } q^{-1}(U) \text{ is open in } X.$$

In other words, q is a quotient map if the topology on Y coincides with the quotient topology via q.

Whenever we have a quotient map $q : X \to Y$, we can identify Y with the quotient space X^* of X by the equivalence relation induced by q. More precisely, we have the following proposition.

Proposition 3.103. *Let $q : X \to Y$ be a quotient map. Define $\sim$ on X by $x \sim x'$ iff $q(x) = q(x')$ and let $X^* := X/\sim = \{q^{-1}(y) : y \in Y\}$ endowed with the quotient topology. Then, q induces a homeomorphism $\varphi : X^* \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, two heads, "p"] \arrow[dr, two heads, "q"'] & X^* \arrow[d, "\varphi"] \\ & Y \end{tikzcd}$$

Proof. We first define $\varphi : X^* \to Y$ by $\varphi(q^{-1}(y)) = y$. This is well-defined since $q^{-1}(y) = q^{-1}(y')$ implies $y = y'$. It is easy to see that φ is bijective and the diagram commutes.

Next, we show that φ is continuous. For any open set $U \subseteq Y$, we have that

$$p^{-1}(\varphi^{-1}(U)) = q^{-1}(U)$$

is open since q is continuous. By the definition of the quotient topology, $\varphi^{-1}(U)$ is open in X^* and hence φ is continuous.

Finally, we show that φ^{-1} is continuous. For any open set $V \subseteq X^*$, we have that

$$q^{-1}(\varphi(V)) = p^{-1}(V)$$

is open since p is continuous. By the definition of the quotient map, $\varphi(V)$ is open in Y and hence φ^{-1} is continuous. $\square$

Here is some simple criterion for quotient maps. A map is said to be *open* if the image of every open set is open. Similarly, a map is said to be *closed* if the image of every closed set is closed.

Theorem 3.104. *Let $q : X \to Y$ be a surjective continuous map. If q is open or closed, then q is a quotient map.*

Proof. We only need to show the "only if" direction. Suppose q is open. Let $U \subseteq Y$ be such that $q^{-1}(U)$ (we used surjection here) is open in X. Then, $U = q(q^{-1}(U))$ is open in Y since q is open. The case when q is closed is similar. $\square$

Example 3.105. Let $X = [0, 2\pi]$ with the usual topology from $\mathbb{R}$. Let $Y = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ with the subspace topology from $\mathbb{R}^2$. Define

$$p : X \to Y, \quad p(t) = (\cos t, \sin t).$$

Then, p is surjective and continuous. p is not open unfortunately: $[0, a)$ will not be mapped to an open set. But p is closed. Later when we have more tools, this can be proved easily. But now, let's use techniques we have to prove it.

Proof that p is closed. Let $F \subset [0, 2\pi]$ be closed. We show $p(F)$ is closed in S^1. Since $S^1 \subset \mathbb{R}^2$ is a metric space, it suffices to use sequences.

Take any sequence $(y_n) \subset p(F)$ with $y_n \to y \in S^1$. For each n choose $t_n \in F$ such that $p(t_n) = y_n$. The sequence (t_n) lies in the bounded interval $[0, 2\pi]$, so by Bolzano–Weierstrass it has a convergent subsequence $t_{n_k} \to t \in [0, 2\pi]$. Because F is closed and each $t_{n_k} \in F$, we have $t \in F$. By continuity of p,

$$p(t_{n_k}) \longrightarrow p(t).$$

But $p(t_{n_k}) = y_{n_k} \to y$, hence $y = p(t) \in p(F)$. Thus every limit of a convergent sequence in $p(F)$ lies in $p(F)$; so $p(F)$ is closed in S^1.

Therefore p sends closed subsets of $[0, 2\pi]$ to closed subsets of S^1, i.e. p is a closed map. Since p is surjective and continuous, it is a quotient map. $\square$

In this way, we can identify S^1 with the quotient space obtained by attaching end points of an interval $[0, 2\pi]/(0 \sim 2\pi)$.

Example 3.106. All quotient topology examples above are quotient maps when the targets are endowed with subspace topologies. In fact, all maps are closed which we can prove later when we learn about compactness. Right now, we can just use the sequence argument as in the previous example.

Example 3.107. Consider the subspace

$$X = \{(x, y) \in \mathbb{R}^2 : xy = 1\} \ \cup\ \{(0, 0)\} \subset \mathbb{R}^2, \qquad q : X \longrightarrow \mathbb{R},\ q(x, y) = x.$$

The map q is surjective: for $x \neq 0$ choose $y = 1/x$, and $q(0, 0) = 0$. It is continuous as the restriction of the projection $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$.

However q is not a quotient map. Indeed $\{(0, 0)\} = B_\varepsilon(0, 0) \cap X$ for small $\varepsilon > 0$, so $\{(0, 0)\}$ is open in X. Thus

$$q^{-1}(\{0\}) = \{(0, 0)\} \text{ is open in } X,$$

but $\{0\}$ is not open in $\mathbb{R}$. Hence the "only if" direction fails and q is not a quotient map.

The following theorem is analogous to the universal property of product topology.

Theorem 3.108. *Let $q : X \to Y$ be a quotient map and let Z be a topological space. Let $g : X \to Z$ be such that g is constant on each fiber $q^{-1}(\{y\})$. This induces a function $f : Y \to Z$ such that $f \circ q = g$. Then*

1. *f is continuous iff g is continuous.*
2. *f is a quotient map iff g is a quotient map.*

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "f"'] & Z \end{tikzcd}$$

Proof. (1) ($\Rightarrow$) Composition of continuous maps is continuous.

($\Leftarrow$) Suppose g is continuous. Let $V \subseteq Z$ be open. Then

$$(f \circ q)^{-1}(V) = q^{-1}\big(f^{-1}(V)\big)$$

is open in X. Since q is a quotient map, $f^{-1}(V)$ is open in Y; hence f is continuous.

(2) ($\Rightarrow$) Suppose g is a quotient map. Let $U \subseteq Z$ be such that $f^{-1}(U)$ is open in Y. Then

$$q^{-1}(f^{-1}(U)) = (f \circ q)^{-1}(U) = g^{-1}(U)$$

is open in X. Since g is a quotient map, U is open in Z. Hence f is a quotient map.

($\Leftarrow$) Suppose f is a quotient map. Let $U \subseteq Z$ be such that $g^{-1}(U)$ is open in X. Then

$$q^{-1}(f^{-1}(U)) = (f \circ q)^{-1}(U) = g^{-1}(U)$$

is open in X. Since q is a quotient map, $f^{-1}(U)$ is open in Y. Since f is a quotient map, U is open in Z. Hence g is a quotient map. The same proof shows that the composition of two quotient maps is a quotient map. $\square$

3.9 Homotopy

Example 3.109. $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. How about $\mathbb{R}^2$ vs $\mathbb{R}^3$?

We would like to answer this question to some extent. In the process of answering this question, we will introduce a very important concept in topology, called homotopy.

Definition 3.110. Let f and g be continuous maps from a topological space X to a topological space Y. We say that f and g are *homotopic* if there exists a continuous map $H\colon X\times[0,1]\to Y$ such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$ for all $x\in X$. H is called a *homotopy* between f and g. We write $f\simeq g$ to indicate that f and g are homotopic.

Example 3.111. Any continuous map $f\colon X\to Y$ is homotopic to itself via the constant homotopy $H(x,t)=f(x)$.

Example 3.112. In $\mathbb{R}^n$ the identity map $f(x)=x$ is homotopic to the constant map $g(x)=0$ via the *straight–line homotopy*

$$H(x,t)=(1-t)x.$$

Example 3.113. More generally, if Y is a convex subset of some $\mathbb{R}^n$ and $f,g\colon X\to Y$ are continuous, then

$$H(x,t)=(1-t)f(x)+tg(x)$$

defines a homotopy $f\simeq g$. Thus any two maps into a convex (in particular, contractible) space are homotopic. The target matters: regarded as maps $S^1\to D^2$ the identity and any constant map are homotopic, but as maps $S^1\to S^1$ they need not be. This is in fact quite nontrivial and we will return to it later.

Proposition 3.114. *Homotopy is an equivalence relation on $C(X,Y)$, the set of continuous maps $X\to Y$.*

Proof. Reflexivity: $f\simeq f$ via the constant homotopy $H(x,t)=f(x)$.

Symmetry: if H is a homotopy $f\simeq g$, then $H'(x,t)=H(x,1-t)$ is a homotopy $g\simeq f$.

Transitivity: if H_1 is a homotopy $f\simeq g$ and H_2 a homotopy $g\simeq h$, define

$$H(x,t)=\begin{cases}H_1(x,2t) & 0\le t\le \frac{1}{2},\\ H_2(x,2t-1) & \frac{1}{2}\le t\le 1.\end{cases}$$

Continuity follows from the pasting lemma (here we used the fact that the product of two closed sets is closed) because the two pieces agree at $t=\frac{1}{2}$. Then H is a homotopy $f\simeq h$.

□

Proposition 3.115 (Composites of homotopic maps are homotopic)**.** *If $f_1,g_1\colon X\to Y$ and $f_2,g_2\colon Y\to Z$ with $f_1\simeq g_1$ and $f_2\simeq g_2$, then $f_2\circ f_1\simeq g_2\circ g_1$.*

Proof. Let $H_1 \colon X \times I \to Y$ be a homotopy $f_1 \simeq g_1$ and $H_2 \colon Y \times I \to Z$ a homotopy $f_2 \simeq g_2$. Then $f_2 \circ H_1$ is a homotopy $f_2 \circ f_1 \simeq f_2 \circ g_1$. Define $H(x,t) = H_2(g_1(x),t)$; this is continuous as a composite and gives a homotopy $f_2 \circ g_1 \simeq g_2 \circ g_1$. Transitivity yields $f_2 \circ f_1 \simeq g_2 \circ g_1$. □

With the notion of homotopy between functions, we can define homotopy equivalence between spaces.

Definition 3.116. Two spaces X and Y are *homotopy equivalent* if there exist continuous maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. In this case f and g are homotopy inverses and we write $X \simeq Y$.

Remark 3.117. Homotopy equivalence is an equivalence relation on spaces. Any homeomorphism is a homotopy equivalence (take $g = f^{-1}$). The converse fails; see the example below.

One special case of homotopy equivalence is when one of the spaces is a point.

Definition 3.118. A space X is *contractible* if the identity map id_X is homotopic to a constant map $c_{x_0} \colon X \to X$ sending every point to x_0. Equivalently, there exists $H \colon X \times I \to X$ with $H(x,0) = x$ and $H(x,1) = x_0$ for all x.

Theorem 3.119. *A space X is contractible iff it is homotopy equivalent to a one–point space.*

Proof. If X is contractible pick $x_0 \in X$ and set $Y = \{x_0\}$. Let $j \colon Y \hookrightarrow X$ be inclusion and $c \colon X \to Y$ the constant map. Then $c \circ j = id_Y$ and $j \circ c = c \simeq id_X$, so $X \simeq Y$.

Conversely, suppose $f \colon X \to Y$ and $g \colon Y \to X$ exhibit a homotopy equivalence where Y is a point. Then $g \circ f$ is constant and $g \circ f \simeq id_X$, giving a homotopy from id_X to a constant map, so X is contractible. □

Example 3.120. Every convex subset of $\mathbb{R}^n$ (in particular $\mathbb{R}^n$ and any disk D^n) is contractible via the straight–line homotopy $H(x,t) = (1-t)x + tx_0$.

Example 3.121. $\mathbb{R}^n$ is contractible but $\mathbb{R}^n$ is not homeomorphic to a point.

Example 3.122. S^1 is not contractible. We will prove this later.

The contractible case is the trivial case of homotopy equivalence. More generally, we can often find a smaller subspace which is homotopy equivalent to the whole space. This is made precise by the following definitions.

Definition 3.123. Let $A \subseteq X$. A map $r \colon X \to A$ is a *retraction* if $r(a) = a$ for all $a \in A$ (equivalently $r \circ i = \mathrm{id}_A$ where $i \colon A \hookrightarrow X$ is inclusion).

Definition 3.124. A subspace $A \subseteq X$ is a *deformation retract* of X if there is a retraction $r \colon X \to A$ such that there is a homotopy $H \colon X \times I \to X$ with $H(x,0) = x$, $H(x,1) = r(x)$, and $H(a,t) = a$ for all $a \in A$, $t \in I$. We call H a *deformation retraction.*

Proposition 3.125. *If A is a deformation retract of X then $A \simeq X$. Thus a deformation retract gives a special homotopy equivalence (with inverse given by the retraction).*

Proof. Let $r\colon X \to A$ be a retraction and $i\colon A \hookrightarrow X$ inclusion with $i \circ r \simeq \mathrm{id}_X$. Then $r \circ i = \mathrm{id}_A$, so i and r are homotopy inverses. $\square$

Example 3.126.

1. The circle S^1 is a deformation retract of any annulus $A = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq R\}$ with the retraction $r(x) = x/\|x\|$ and the homotopy $H(x,t) = (1-t)x + tr(x)$.
2. $\mathbb{R}^n \setminus \{0\}$ deformation retracts onto S^{n-1}: the retraction is $r(x) = x/\|x\|$ and the homotopy is $H(x,t) = (1-t)x + tr(x)$.

Example 3.127. Not every retraction is a deformation retraction. In fact, for any X and $x_0 \in X$, the constant map $c_{x_0}\colon X \to \{x_0\}$ is a retraction if $\{x_0\}$ is a retract of X, but it is a deformation retraction only if X is contractible.

Theorem 3.128. *There is no retraction $D^2 \to S^1$.*

This can be proved using algebraic topology (fundamental group) or using homology. We will give a proof without these later.

Now we are ready to answer the question whether $\mathbb{R}^2$ and $\mathbb{R}^3$ are homeomorphic.

Theorem 3.129. *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.*

Proof. Suppose on the contrary there is a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Then f restricts to a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. But $\mathbb{R}^2 \setminus \{0\} \simeq S^1$ and $\mathbb{R}^3 \setminus \{f(0)\} \simeq S^2$. So $S^1 \simeq S^2$. We will show this is not true.

We will use the following lemmas.

Lemma 3.130. *The identity map on S^1 is not homotopic to a constant map. (This proves S^1 is not contractible.)*

Lemma 3.131. *Every map $S^1 \to S^2$ is homotopic to a constant map.*

Now suppose $S^1 \simeq S^2$ via maps $f\colon S^1 \to S^2$ and $g\colon S^2 \to S^1$. Then, we know that f is homotopic to a constant map $c : S^1 \to S^2$. So $g \circ f : S^1 \to S^1$ is homotopic to $g \circ c$, which is a constant map. But $g \circ f \simeq \mathrm{id}_{S^1}$, which contradicts the first lemma.

$\square$

The proof of the first lemma is as follows.

Proof of Lemma 3.130. We need the following lemma.

Lemma 3.132 (Lifting lemma)**.** *Let $p : \mathbb{R} \to S^1$ be $p(t) = e^{it}$. If $f : S^1 \to S^1$ is homotopic to a constant map, than f lifts, i.e., there exists a continuous $\tilde{f} : S^1 \to \mathbb{R}$ with $p \circ \tilde{f} = f$.*

Proof. We first show that if $g : S^1 \to S^1$ lifts and $h : S^1 \to S^1$ satisfies $h(x) \neq -g(x)$ for all $x \in S^1$, then h lifts. Indeed, $h(x)/g(x) \in S^1$ (viewed as complex numbers) is never -1, so we can write $h(x)/g(x) = e^{i\psi(x)}$ with a unique continuous $\psi(x) \in (-\pi, \pi)$. If $\tilde{g}$ is a lift of g, i.e., $g(x) = e^{i\tilde{g}(x)}$, then $h(x) = g(x)e^{i\psi(x)} = e^{i(\tilde{g}(x)+\psi(x))}$, so $\tilde{h}(x) = \tilde{g}(x) + \psi(x)$ is a lift of h.

Now, let $H : S^1 \times I \to S^1$ be a homotopy from the constant map $c(z) = 1$ to f. Equip S^1 with the chord metric $d(u, v) = |u - v|$ (so $d(u, v) = 2$ iff $v = -u$). Since H is uniformly continuous on the compact set $S^1 \times I \subset \mathbb{R}^3$, choose $\delta > 0$ so that

$$|t - s| < \delta \implies d\big(H(z, t), H(z, s)\big) < 2 \quad \text{for all } z \in S^1.$$

Pick $0 = t_0 < t_1 < \cdots < t_N = 1$ with $t_{j+1} - t_j < \delta$.

The map $H(\,\cdot\,, 0) = c_1$ lifts to $\tilde{H}_N : S^1 \to \mathbb{R}$ by $\tilde{H}_N(z) \equiv 0$.

Inductive step. Suppose H_j lifts. For any z, the points $H(z, t_j) \neq H(z, t_{j+1})$ since $d(H(z, t_j), H(z, t_{j+1})) < 2$, so $H(z, t_{j+1})$ lifts as well.

By induction we have that $f = H(\cdot, 1)$ lifts. □

Now, suppose id_{S^1} is homotopic to a constant map. Then, by the lifting lemma, id_{S^1} lifts to a continuous map $\tilde{f} : S^1 \to \mathbb{R}$ with $p \circ \tilde{f} = \mathrm{id}_{S^1}$. The lift $\tilde{f}$ must be injective because id_{S^1} is injective. This would violate the intermediate value theorem: choose $\alpha \in (\phi(-1), \phi(1))$ (here we assume $\phi(-1) < \phi(1)$). Then both arcs in S^1 from -1 to 1 must contain a point z with $\phi(z) = \alpha$, contradicting injectivity. □

Proof of Lemma 3.131. Let $f : S^1 \to S^2$ be continuous. Pick $p \in S^2 \setminus f(S^1)$. ($f$ could be surjective but let's ignore this case and it can be fixed using some more arguments).

Observe that $S^2 - \{p\} \cong \mathbb{R}^2$. Let $g : S^2 - \{p\} \to \mathbb{R}^2$ be a homeomorphism. Then, $g \circ f : S^1 \to \mathbb{R}^2$ is continuous. Since $\mathbb{R}^2$ is convex, $g \circ f$ is homotopic to a constant map via the straight–line homotopy. Composing with g^{-1} gives a homotopy from f to a constant map. More precisely, if $H_1 : S^1 \times I \to \mathbb{R}^2$ is a homotopy from $g \circ f$ to a constant map, then $H_2 = g^{-1} \circ H_1 : S^1 \times I \to S^2 - \{p\} \hookrightarrow S^2$ is a homotopy from f to a constant map.

□