Brouwer Fixed Point Theorem

Now, we introduce some interesting results and prove some special cases using the fact that S^1 is not contractible.

Theorem 3.133 (No retraction)**.** *There is no retraction $D^n \to S^{n-1}$ for $n \geq 1$.*

Proof. We prove the case when $n = 2$. Suppose there is a retraction $r : D^2 \to S^1$. Then, we define $H : S^1 \times I \to S^1$ as follows:

$$H(x, t) = r(tx).$$

Then, $H(x, 0) = r(0)$ is a constant map and $H(x, 1) = r(x) = x$. So H is a homotopy from the identity map on S^1 to a constant map. This contradicts Lemma 3.130. □

Proof. If $r\colon D^n \to S^{n-1}$ were a retraction, compose any map $S^{n-1} \to D^n$ extending $\mathrm{id}_{S^{n-1}}$ with r to obtain a fixed–point–free self map of D^n, contradicting Brouwer. □

Corollary 3.134 (Brouwer Fixed Point)**.** *For every $n \geq 1$, any continuous map $F\colon D^n \to D^n$ has a fixed point, i.e. there exists $x \in D^n$ with $F(x) = x$.*

Proof. Assume F has no fixed point. For each $x \in D^n$ draw the ray starting at $F(x)$ through x and let $r(x)$ be the first intersection of that ray with $S^{n-1} = \partial D^n$. This gives a continuous map $r\colon D^n \to S^{n-1}$ which restricts to the identity on S^{n-1}, hence a retraction. But S^{n-1} is not a retract (indeed not a deformation retract) of D^n (proved earlier using homotopy ideas). Contradiction. □

Homotopy equivalence = embedding + retraction

If there is one thing that you need to remember, that is homotopy equivalence is basically embedding + retraction.

First of all, we introduce the notion of embeddings.

Definition 3.135 (Embedding)**.** A continuous map $f : X \to Y$ is an *embedding* if it is a homeomorphism onto its image $f(X)$ (with the subspace topology inherited from Y). In particular, f is injective.

Definition 3.136. Let X and Y be two disjoint topological spaces. Then, the *disjoint union* of X and Y is the topological space

$$X \sqcup Y$$

whose underlying set is the disjoint union of sets X and Y and whose topology is given by

$$U \subseteq X \sqcup Y \text{ is open} \iff U \cap X \text{ is open in } X \text{ and } U \cap Y \text{ is open in } Y.$$

Equivalently, the topology is given by

$$\tau_{X \sqcup Y} = \{U \sqcup V \mid U \in \tau_X, V \in \tau_Y\}.$$

Definition 3.137 (Mapping cylinder)**.** Given a continuous map $f : X \to Y$, the *mapping cylinder* of f is

$$M_f = (X \times [0,1]) \sqcup Y \big/ \sim$$

where $(x, 1) \sim f(x)$ for all $x \in X$. Write $[x, t]$ for the class of (x, t) and $[y]$ for $y \in Y$. There are canonical embeddings

$$i_X : X \hookrightarrow M_f, \qquad x \mapsto [x, 0], \qquad \text{and} \qquad i_Y : Y \hookrightarrow M_f, \qquad y \mapsto [y].$$

There is a deformation retraction $H : M_f \times I \to M_f$ onto $i_Y(Y)$ given by $H([x,t], s) = [x, (1-s)t + s]$ and $H([y], s) = [y]$, so M_f deformation retracts onto Y.

Lemma 3.138. *For any continuous map $f : X \to Y$, the map $H : M_f \times I \to M_f$ defined by $H([x,t], s) := [x, (1-s)t + s]$ and $H([y], s) := [y]$ is a deformation retraction of M_f onto $i_Y(Y)$.*

Lemma 3.139. *$f : X \to Y$ is a homotopy equivalence if and only if $i_X(X)$ is a deformation retract of M_f.*

Theorem 3.140 (Common deformation–retract envelope)**.** *For topological spaces X and Y, the following are equivalent:*

1. *X and Y are homotopy equivalent.*
2. *There exists a space Z and embeddings $i_X : X \hookrightarrow Z$, $i_Y : Y \hookrightarrow Z$ such that $i_X(X)$ and $i_Y(Y)$ are each* deformation retracts *of Z (possibly via two different homotopies).*

Proof. (2) $\Rightarrow$ (1): If Z deformation retracts onto $i_X(X)$ and onto $i_Y(Y)$, then $Z \simeq X$ and $Z \simeq Y$, hence $X \simeq Y$ by transitivity.

(1) $\Rightarrow$ (2): Suppose $f : X \to Y$ is a homotopy equivalence. Form the mapping cylinder M_f, which admits deformation retractions onto Y and X. $\square$