4 Continuity and Convergence

4.1 Convergence in Topological Spaces

We will try to answer the following questions:

1. Can we define sequential convergence in general topological spaces?
2. If so, does this convergence describe the notion of topology?

In metric space setting, we have the notion of convergence of a sequence (in the language of open balls):

Proposition 4.1. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space X converges to a point $x \in X$ if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$.*

This motivates the following definition in general topological spaces:

Definition 4.2. A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to converge to a point $x \in X$ if for every neighbourhood U of x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Problem 4.3. A sequence (x_n) in a metric space converges in the metric sense if and only if it converges in the topological sense defined above.

Proof. ($\Rightarrow$) Let (x_n) be a sequence in a metric space (X, d) converging to $x \in X$ in the metric sense. For every neighborhood U of x, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Then, there exists N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In particular, $x_n \in B_\epsilon(x) \subset U$ for all $n \geq N$. So we have $x_n \to x$ in the topological sense.

($\Leftarrow$) Let (x_n) be a sequence in a metric space (X, d) converging to $x \in X$ in the topological sense. For every $\epsilon > 0$, the open ball $B_\epsilon(x)$ is a neighbourhood of x, so there exists N such that $x_n \in B_\epsilon(x)$ for all $n \geq N$. This means that $d(x_n, x) < \epsilon$ for all $n \geq N$. Hence, (x_n) converges to x in the metric sense. $\square$

Proposition 4.4. *Let X be a topological space. For $A \subseteq X$ and $x \in X$, if there is a sequence $(a_n) \subseteq A$ with $a_n \to x$, then $x \in \overline{A}$.*

Proof. Let U be any neighbourhood of x. Then there exists N such that $a_n \in U$ for all $n \geq N$. In particular, $a_N \in A \cap U$, so $A \cap U \neq \emptyset$. Since U was arbitrary, we have $x \in \overline{A}$. $\square$

We know the converse is true in metric spaces. Is it true in general topological spaces?

Example 4.5. Let $X = \mathbb{R}$ be endowed with the *countable complement topology*: a set $U \subseteq X$ is open iff $X \setminus U$ is countable or $U = \emptyset$.

- Any sequence (x_n) converging to x is eventually equal to x. Indeed, we let $B = \{x_n : x_n \neq x\}$ and hence $U = X \setminus B$ is an open neighborhood of x. Then, there exists N such that $x_n \in U$ for all $n \geq N$, i.e., $x_n \notin B$ for all $n \geq N$. This means that $x_n = x$ for all $n \geq N$.

- The closed sets are exactly the countable sets and X itself. Hence, if $A \subsetneq X$ is uncountable, then $\overline{A} = X$.

Thus $\overline{[0,1]} = \mathbb{R}$, but there is no sequence in $[0,1]$ converging to 2.

Let's see another example.

Example 4.6. Let $X = \mathbb{R}^{\mathbb{R}}$ be endowed with the product topology. We consider

$$E = \{f \in \mathbb{R}^{\mathbb{R}} : f(x) = 0 \text{ or } 1 \text{ and } f(x) = 0 \text{ only finitely often}\}.$$

Let $\mathbf{0}$ denote the all 0 function in X. Note that any basis element of the product topology containing $\mathbf{0}$ is of the form

$$V = \{g \in \mathbb{R}^{\mathbb{R}} : g(x_i) \in U_i \text{ for } i = 1, \ldots, n\}$$

for some $x_1, \ldots, x_n \in \mathbb{R}$ and open sets $U_i \ni \mathbb{R}$ containing 0. We now define $g : \mathbb{R} \to \mathbb{R}$ as follows: $g(x_i) = 0$ for $i = 1, \ldots, n$ and $g(x) = 0$ otherwise. Then, $g \in E \cap V$ and hence $E \cap V \neq \emptyset$. Hence, $\mathbf{0} \in \overline{E}$.

Now let (f_n) be a sequence in E such that each f_n is 0 on a finite set $A_n \subset \mathbb{R}$. If (f_n) converges to f, then f can be zero at most on the countable set $\bigcup_n A_n$. Suppose otherwise $f(x) = 0$ for some $x \notin \bigcup_n A_n$. Pick the basic open neighbourhood

$$U = \{g \in \mathbb{R}^{\mathbb{R}} : g(x) \in (-1, 1)\}.$$

Then, there exists N such that $f_n \in U$ for all $n \geq N$. In particular, $f_n(x) \in (-1, 1)$ for all $n \geq N$. This implies that $f_n(x) = 0$ for all $n \geq N$, contradicting the choice of x.

Hence, there is no sequence in E converging to $\mathbf{0}$.

In metric spaces, the limit of a convergent sequence is unique. Is this true in general topological spaces?

Example 4.7. Let X be endowed with the trivial topology $\tau = \{\emptyset, X\}$. Then the constant sequence (x_n) with $x_n = a \in X$ for all $n \in \mathbb{N}$ converges to every point in X.

Example 4.8. Let $X = \{a, b\}$ with the topology $\tau = \{\emptyset, \{a\}, X\}$. Then the constant sequence (x_n) with $x_n = a$ for all $n \in \mathbb{N}$ converges to a and b.

In general, the limit of a convergent sequence may not be unique. We will need to give some additional conditions on the topology to ensure uniqueness of limits.

Finally, let's talk about continuity.

Proposition 4.9. *Let $f : X \to Y$ be continuous at x. Then, whenever $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.*

Proof. Let U be any neighbourhood of $f(x)$ in Y. By continuity of f at x, there exists a neighbourhood V of x such that $f(V) \subseteq U$. Since $x_n \to x$, there exists N such that for all $n \geq N$, $x_n \in V$. Hence, for all $n \geq N$, $f(x_n) \in U$, which shows that $f(x_n) \to f(x)$. $\square$

The converse is not true in general. We will see an example below. But before that, let's see an interesting consequence.

Theorem 4.10. *Let $X = \Pi X_\alpha$ be a product of topological spaces X_α. A sequence (x_n) in X converges to $x \in X$ if and only if for every α, the sequence $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ in X_α.*

Proof. ($\Rightarrow$) This follows from the continuity of the projection maps π_α.

($\Leftarrow$) Let (x_n) be a sequence in X such that for every α, the sequence $(\pi_\alpha(x_n))$ converges to $\pi_\alpha(x)$ in X_α. Let U be a subbasis neighbourhood of x in X of the form

$$U = \pi_\alpha^{-1}(V) = V \times \prod_{\beta \neq \alpha} X_\beta,$$

where α is an index and V is a neighbourhood of $\pi_\alpha(x)$ in X_α. Since $\pi_\alpha(x_n) \to \pi_\alpha(x)$, there exists N such that $\pi_\alpha(x_n) \in V$ for all $n \geq N$. Then, for all $n \geq N$, we have $x_n \in U$. Hence, (x_n) converges to x in X. $\square$

Example 4.11 (Sequential continuity does not imply continuity)**.** Let $X = \{0,1\}^{\mathbb{R}}$ with the product topology. For $x \in X$ put $\operatorname{supp}(x) = \{t \in \mathbb{R} : x(t) = 1\}$. Let $S = \{0,1\}$ with the topology $\{\varnothing, \{1\}, \{0,1\}\}$. Define

$$f : X \to S, \qquad f(x) = \begin{cases} 0, & \text{if } \operatorname{supp}(x) \text{ is countable,} \\ 1, & \text{if } \operatorname{supp}(x) \text{ is uncountable.} \end{cases}$$

Note: in S, a sequence $y_n \to 1$ iff it is eventually 1, while every sequence converges to 0.

(1) f preserves sequential convergence. Let $x_n \to x$ in X. By theorem above, we have $x_n(t) \to x(t)$ for each $t \in \mathbb{R}$.

- If $\operatorname{supp}(x)$ is countable, then $f(x) = 0$, and every sequence in S converges to 0, so $f(x_n) \to 0 = f(x)$.
- If $\operatorname{supp}(x)$ is uncountable, then for each $t \in \operatorname{supp}(x)$ there exists N_t such that $x_n(t) = 1$ for all $n \geq N_t$. Set $V_m = \{t \in \operatorname{supp}(x) : N_t \leq m\}$. The V_m increase and $\bigcup_{m=1}^{\infty} V_m = \operatorname{supp}(x)$, hence some V_{m_*} is uncountable. For all $n \geq m_*$ and any $t \in V_{m_*}$, since $N_t \leq m_* \leq n$, we have $x_n(t) = 1$. This implies that $t \in \operatorname{supp}(x_n)$. Hence, we have $V_{m_*} \subseteq \operatorname{supp}(x_n)$, so $\operatorname{supp}(x_n)$ is uncountable and $f(x_n) = 1$. Thus $f(x_n) \to 1 = f(x)$.

(2) f is not continuous. Pick $y \in X$ with uncountable support (so $f(y) = 1$). Continuity at y would give a basic neighbourhood U of y with $U \subseteq f^{-1}(\{1\})$. But any basic neighbourhood of y is of the form

$$U = \prod_{i \in F} U_i \times \prod_{t \in \mathbb{R} \setminus F} \{0, 1\}$$

for some finite $F \subset \mathbb{R}$. Hence, we define z by $z|_F = y|_F$ and $z(t) = 0$ for $t \notin F$. Then $z \in U$ and $\operatorname{supp}(z) \subseteq F$ is finite, so $f(z) = 0$. Hence $U \not\subseteq f^{-1}(\{1\})$, a contradiction. Therefore f is not continuous.

4.2 Nets

How do we fix this? We need a more general notion of convergence.

Note that a sequence (x_n) in X is in fact a function from $x : \mathbb{N} \to X$. Instead of just using $\mathbb{N}$ as the index set, we can use more general index sets.

Definition 4.12. A set Λ is a *directed set* if it is a non-empty set with a relation $\leq$ such that

- reflexive: for all $\lambda \in \Lambda$, $\lambda \leq \lambda$,
- transitive: for all $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$, if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$,
- directed: for all $\lambda_1, \lambda_2 \in \Lambda$, there exists $\mu \in \Lambda$ such that $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$.

This relation $\leq$ is called a direction on Λ.

Remark 4.13. Note a direction may not be a partial order, i.e., it may not be antisymmetric. For example, let $\Lambda = \{a, b\}$ with $a \leq b$ and $b \leq a$. This is a directed set but not a poset because $a \neq b$.

Example 4.14. $\mathbb{N}$ with the standard order (which is obviously a direction) is a directed set.

Definition 4.15. A net in a topological space X is a function $P \colon \Lambda \to X$ where Λ is a directed set. We often write $P(\lambda) = x_\lambda$ and write the net $(x_\lambda)_{\lambda \in \Lambda}$ or (x_λ).

Definition 4.16. We say that a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to a point $x \in X$ if for every neighbourhood U of x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

Remark 4.17. Note that if we take $\Lambda = \mathbb{N}$ with the standard order, then the notion of convergence of nets reduces to the notion of convergence of sequences.

Example 4.18. Let $\mathcal{P}$ denote the collection of all finite partitions of the closed interval $[a, b]$ into closed subintervals. In other words, each $P \in \mathcal{P}$ has the form $P = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$ where $a = x_0 < x_1 < \cdots < x_n = b$.

This becomes a directed set if we define the direction by refinement: for two partitions $P_1, P_2 \in \mathcal{P}$, we say that $P_1 \le P_2$ if P_2 is a refinement of P_1 (i.e., every subinterval in P_1 is a union of subintervals in P_2).

Now, consider a function $f : [a, b] \to \mathbb{R}$. Define a net $(R_L(P))_{P\in\mathcal{P}}$ as follows: for each partition $P = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$, let

$$R_L(P) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}).$$

This net is called the *lower Riemann sum* net of f.

Similarly, we can define the *upper Riemann sum* net $(R_U(P))_{P\in\mathcal{P}}$ by

$$R_U(P) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}).$$

If both nets converge to the same limit L, then we say that f is Riemann integrable on $[a, b]$ with integral L.

Example 4.19. One typical example of a net is the following: let X be a topological space and $x \in X$. Let Λ be the set of all neighbourhoods of x and define the direction on Λ by $U \le V$ iff $V \subseteq U$ (reverse inclusion). You should check that this is indeed a directed set. Pick a point $x_U \in U$ for each neighbourhood U of x. Then the net $(x_U)_{U\in\Lambda}$ converges to x. To see this, let V be any neighbourhood of x, then for all $U \ge V$, we have $x_U \in U \subseteq V$. This shows that $x_U \to x$.

Theorem 4.20. *If $E \subset X$, then $x \in \overline{E}$ iff there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in E such that $x_\lambda \to x$.*

Proof. If $x \in \overline{E}$, then every neighbourhood U of x intersects E. Choose $x_U \in E \cap U$ for each neighbourhood U of x. Then the net $(x_U)_{U\in\Lambda}$ is a net converging to x.

Conversely, if there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in E such that $x_\lambda \to x$, then for every neighbourhood U of x, there exists λ_0 such that $x_\lambda \in U$ for all $\lambda \ge \lambda_0$. In particular, $x_{\lambda_0} \in E \cap U$, so $E \cap U \ne \emptyset$. Since U was arbitrary, we have $x \in \overline{E}$. □

Theorem 4.21. *Let $f : X \to Y$. Then f is continuous at $x \in X$ iff for every net $(x_\lambda)_{\lambda\in\Lambda}$ in X such that $x_\lambda \to x$, we have $f(x_\lambda) \to f(x)$.*

Proof. ($\Rightarrow$) Given a neighborhood V of $f(x)$ in Y, by continuity of f at x, there exists a neighbourhood U of x such that $f(U) \subseteq V$. Since $x_\lambda \to x$, there exists λ_0 such that $x_\lambda \in U$ for all $\lambda \ge \lambda_0$. Hence, for all $\lambda \ge \lambda_0$, we have $f(x_\lambda) \in V$, which shows that $f(x_\lambda) \to f(x)$.

($\Leftarrow$) Suppose f is not continuous at x. Then there exists a neighbourhood V of $f(x)$ such that for every neighbourhood U of x, there exists $x_U \in U$ with $f(x_U) \notin V$. Let Λ be the set of all neighbourhoods of x directed by reverse inclusion. Then the net $(x_U)_{U\in\Lambda}$ converges to x, but $f(x_U) \notin V$ for all $U \in \Lambda$. Hence, $f(x_U)$ does not converge to $f(x)$. □

5 Separation and countability

We see that a general topological space can behave weirdly compared to metric spaces. We want to impose some additional conditions on topological spaces to make them "nicer" and closer to metric spaces. Eventually, we would like to answer the following question: when can a topological space be embedded into a metric space? Of course not all of them as we've seen previously. We will see that the answer is related to separation axioms and countability axioms.

5.1 Countability

Recall the notion of neighbourhoods.

Definition 5.1 (Neighbourhood)**.** Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A set $B \in \mathcal{T}$ is a *(open) neighbourhood* of x if $x \in B$.

Definition 5.2. A space X has a *countable basis at a point* $x \in X$ if there is a countable collection of neighbourhoods $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ of x such that for every neighbourhood V of x, there exists n with $B_n \subseteq V$.

We call X a *first countable space* if it has a countable basis at each point.

Example 5.3. Let X be equipped with the discrete topology. Then X is first countable: for each $x \in X$, the collection $\{\{x\}\}$ is a countable basis at x.

Example 5.4. A metric space (X, d) is first countable: for each $x \in X$, the collection $\{B_{1/n}(x) : n \in \mathbb{N}\}$ is a countable basis at x.

Remark 5.5. For a general first countable space, we can always assume that $B_{n+1} \subseteq B_n$ for all n. This is because we can replace B_n by $\bigcap_{i=1}^n B_i$.

Let's see some non-examples of first countable spaces.

Example 5.6. Let $X = \mathbb{R}$. Then, the complement finite topology on X is not first countable. To see this, suppose there is a countable basis of neighbourhoods $\{B_n : n \in \mathbb{N}\}$ at some point $x \in X$. Since each B_n is open, we have $X \setminus B_n$ is finite for each n. Let $E = x \cup \bigcup_{n \in \mathbb{N}} (X \setminus B_n)$. Then, E is countable. Now, pick any $y \in \mathbb{R} \setminus E$ and consider the open set $U = X \setminus \{y\}$. Note that U is a neighbourhood of x. However, for any n, we have $B_n \not\subseteq U$ since $y \in B_n \setminus U$. This contradicts the assumption that $\{B_n : n \in \mathbb{N}\}$ is a basis at x. Hence, X is not first countable.

Example 5.7. Let X be a metric space with at least 2 points. Then, $X^{\mathbb{R}}$ is not first countable with the product topology. To see this, let $g \in X^{\mathbb{R}}$ and suppose there is a countable basis $\{B_n : n \in \mathbb{N}\}$ at g. Each B_n contains a basic open neighbourhood of the form

$$V_n = \{f \in X^{\mathbb{R}} : f(t_{n,i}) \in U_{n,i} \text{ for } i = 1, \ldots, k_n\}$$

for some $t_{n,1}, \ldots, t_{n,k_n} \in \mathbb{R}$ and open sets $X \neq U_{n,i} \ni g(t_{n,i})$.

Let $E = \bigcup_n \{t_{n,1}, t_{n,2}, \ldots, t_{n,k_n}\}$. Then, $E \subset \mathbb{R}$ is countable as a countable union of finite sets. Now, pick $t_0 \in \mathbb{R} \setminus E$ and create

$$V = \pi_{t_0}^{-1}(U_0)$$

where $X \neq U_0$ is an open set containing $g(t_0)$. Note that V is a basic open neighbourhood of g, but it contains none of V_n since $t_0 \notin E$. This also implies that V contains none of B_n. This contradicts the assumption that $\{B_n : n \in \mathbb{N}\}$ is a basis at g. Hence, $X^{\mathbb{R}}$ is not first countable. So $X^{\mathbb{R}}$ is not metrizable.

We have previously seen the following result in the setting of metric spaces (and the net version for general topological spaces). Now, we can generalize it to first countable spaces.

Theorem 5.8. *Let X be a topological space.*

1. *For $A \subseteq X$ and $x \in X$, if there is a sequence $(a_n) \subseteq A$ with $a_n \to x$, then $x \in \overline{A}$. If X is first countable, the converse holds: $x \in \overline{A}$ implies that there exists a sequence $(a_n) \subseteq A$ with $a_n \to x$.*

2. *If $f : X \to Y$ is continuous at x and $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y. If X is first countable, the converse holds: $x_n \to x \Rightarrow f(x_n) \to f(x)$ for all sequences (x_n) in X implies that f is continuous at x.*

Proof. We only prove the converse statement of (1).

(1) Let $x \in \overline{A}$. Since X is first countable, there exists a countable basis of neighbourhoods $\{B_n : n \in \mathbb{N}\}$ at x. We assume that $B_{n+1} \subseteq B_n$. For each n, pick $x_n \in B_n$. Now, we claim that $(x_n) \to x$: for any neighbourhood V of x, there exists N such that $B_N \subseteq V$. Then, for all $n \geq N$, we have $x_n \in B_n \subseteq B_N \subseteq V$.

□

Definition 5.9. Let X be a topological space. We say that X is *second countable* if it has a countable basis.

Example 5.10. Every second countable space is first countable. The converse is not true. For example, consider an uncountable set X with the discrete topology. Then X is first countable but not second countable. This also implies that not every metric space is second countable.

Example 5.11. Let $X = \mathbb{R}$ be endowed with the complement finite topology. Then, X is not second countable since X is not first countable.

Example 5.12. The space $\mathbb{R}$ with the usual topology is second countable: the collection $\{(a, b) : a, b \in \mathbb{Q}, a < b\}$ forms a countable basis.

Similarly, $\mathbb{R}^n$ has a countable basis $\{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{Q}, a_i < b_i\}$.

$\mathbb{R}^{\mathbb{N}}$ with the product topology also has a countable basis: $\{(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots : a_i, b_i \in \mathbb{Q}, a_i < b_i, n \in \mathbb{N}\}$.